ELEMENTARY PRACTICAL MECHANICS

J.M.JAMESON • C.W.BANKS

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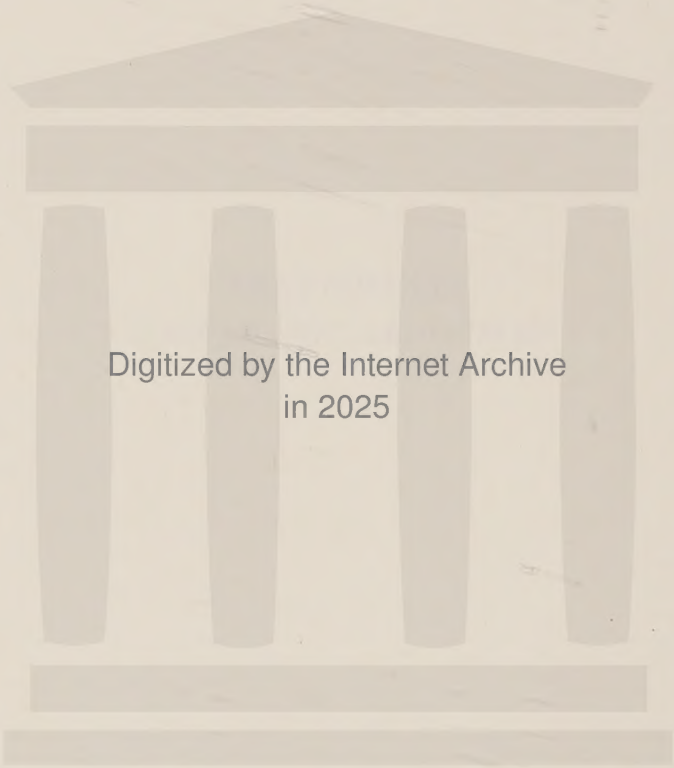

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ELEMENTARY PRACTICAL MECHANICS

BY

THE LATE JOSEPH M. JAMESON, PH.B., PD.D.

Formerly Vice President, Girard College

FOURTH EDITION

BY

CHARLES WILBUR BANKS, C.E.

Formerly Head of Department of Applied Science
Wentworth Institute, Boston, Massachusetts

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PREFACE TO THE FOURTH EDITION

I have long used and admired the late Dr. Jameson's *Elementary Practical Mechanics*. In preparing this revision I have endeavored to retain the spirit and ideals of the original author, and at the same time to increase the usefulness of the book by bringing it up to date. To this end some new material, both text and problems, has been added and some of the original material has been rearranged in such form as teaching experience indicates will be more convenient for the teacher and more readily understood by the student.

Following is a partial list of the revisions.

The standardization of symbols and abbreviations throughout, in general following the recommendations of the American Standards Association. A list of symbols appears in the Appendix.

The use of consistent lettering and arrangement on related diagrams.

The English engineering system of units used uniformly, instead of both English and metric units.

An introductory paragraph at the beginning of each chapter, intended to present very briefly the purpose and value of the material to be studied.

A summary at the end of each chapter in place of the outline for review.

Numerous illustrative problems and examples, with the mathematical steps worked out completely in many of them.

Many new problems for homework and drill.

The rearrangement of the chapters on concurrent forces to permit the graphical and mathematical methods to be studied separately if desired, thereby enabling the student to concentrate on the theory first and on a choice of methods later.

The enlargement of the chapter, Trusses and Other Structures, to include additional methods and additional illustrative examples.

The division of the subject of motion into two chapters, and the division of the subjects of work and energy into two chap-

ters, thus making it more convenient to defer teaching the subjects of accelerated motion, forces producing motion, and energy, until later in the course if so desired.

A new chapter on the elements of heat, added because of the importance of some knowledge of heat energy in the numerous practical applications of mechanics.

Numerous other additions which will be evident throughout the text.

I acknowledge indebtedness to all who have assisted in any way, particularly by contributing criticisms and suggestions, and express sincere appreciation and thanks to them. Especially would I mention Mr. Walter M. Heald, Mr. Mahlon G. Knowles, Mr. Percival R. Allen, and Mr. Henry W. Benson, all of the Department of Applied Science of Wentworth Institute, Boston, Massachusetts; Mr. Raymond L. Freas, of the Pennsylvania State College Extension Services, Conshohocken, Pennsylvania; Mr. Raymond E. Lueder, of the Pennsylvania State College Extension Services, Wilkes-Barre, Pennsylvania; Mr. H. A. Romanowitz, of Ohio Mechanics Institute, Cincinnati, Ohio; and Mr. A. Edwin Wells, of Lowell Textile Institute, Lowell, Massachusetts.

CHARLES WILBUR BANKS

NEWTON HIGHLANDS, MASSACHUSETTS
December 15, 1937

FROM THE PREFACE TO THE FIRST EDITION

The accompanying text is an attempt to express practical mechanics as the science of the processes and structures of everyday life rather than as a series of more or less abstract mathematical demonstrations. It aims to convey not only a knowledge of facts and of fundamental theory, but also some training in ability to apply such knowledge; to equip the student who is to receive no further formal instruction in mechanics with practical information required for business or industrial life, and to establish a foundation and right point of view for the student who is to continue his study along applied lines, which shall enable him to advance without confusion or loss of time in acquiring new methods and a new vocabulary.

It is designed primarily as a text for elementary technical and manual training schools which find the usual texts in elementary mechanics too theoretical or too mathematical for their needs, and which require a more complete and practical course than that furnished by a text in general physics; also as an introductory text for engineering schools to be followed by the more applied courses in mechanism, and in the electrical and mechanical laboratories.

* * * * * * * * *

No apology is offered for the large amount of space devoted to problems. Too many problems in mechanics are scarcely possible. It is believed that the problems in this text will be found carefully selected and well distributed.

* * * * * * * * *

J. M. Jameson

Brooklyn, N. Y.
September, 1909.

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ELEMENTARY PRACTICAL MECHANICS

CHAPTER I

FORCE

Mechanics is the science that treats of forces and the effects of forces. It is therefore necessary for the student, in beginning the study of this subject, to acquire a clear conception of the meaning of the term *force*. This chapter is intended to make clear the significance of the term as used in practical mechanics, together with certain other fundamental principles which will be used in the following chapters.

1. General ideas of force. Everyone is familiar with those commonly known forces associated with muscular action, as when one *pulls* on a rope, or is *pushed* by some one else. In mechanics we must consider many cases of similar "pushes" and "pulls" not originating from muscular action, but through the action of one body of matter upon another, in the various relationships in which these bodies are placed.

FIG. 1. Twenty-ton barge derrick.

Thus, in the hoisting derrick, Fig. 1, the rope is pulling upward upon the weight or "load." In Fig. 2, each column is pushing upward on a roof stringer or beam and thus supporting the load or weight of the roof and any objects resting upon it. If we examine more closely into the conditions existing between the separate bodies in contact, we shall get a clearer idea of what is meant by the term *force*.

2. Action and reaction always equal. Figure 3 shows diagrammatically the conditions for the rope and the load. The pull P of the rope is an action upward applied by the rope to the load;

the weight of the load W is an equal action downward applied by the load to the rope. Figure 4 shows these actions at the top of a column. The load, or weight of the beam and roof, is pushing downward upon the top of the column, as indicated by the action W, and since the roof is supported, the column is pushing upward an equal amount on the beam as indicated by the action R. If we transfer our attention to the foot of the column, as Fig. 5, we note that it is pushing downward with an action W', equal to the combined weight of the column and the load it bears, upon the base on which the column rests, while this base is pushing upward an equal amount, as indicated by the action R'.

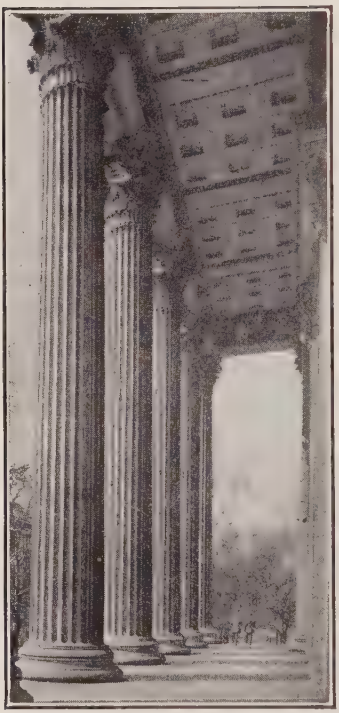


FIG. 2. Stone pillars, or columns, supporting porch roof.

These two actions are always equal in amount and opposite in direction, as P and W, Fig. 3, W and R, Figs. 4 and 5. One is commonly called the **action,** the other the **reaction.** This agrees with Newton's third law of motion. (See Appendix.)

As other illustrations of action and reaction, consider the following examples:

Example 1.

Suppose there is a rope with a boy pulling at each end. Here we have *two* actions, *equal* in amount and *opposite* in direction. *One boy furnishes the action, the other provides the opposing reaction.*

Example 2.

Suppose a rope is tied to a post with a boy pulling at its other end. Here the post is taking the place of the second boy of Example 1, and is exerting the reaction equal and opposite to the pull which the boy is exerting.

Example 3.

Suppose a book is resting upon a table. The earth is exerting a downward pull (gravity) upon the book, causing it to push against the table, and at the same time the table is exerting an upward force against the book which is equal and opposite to the push of the book; if the table failed to exert this upward force, the book would fall.

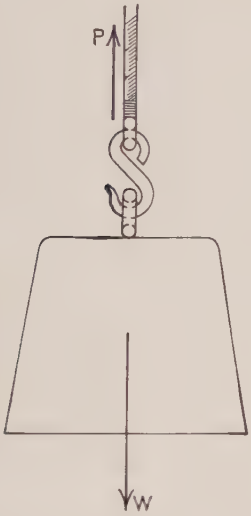


FIG. 3. Action and reaction, rope supporting a load. $P = W$.

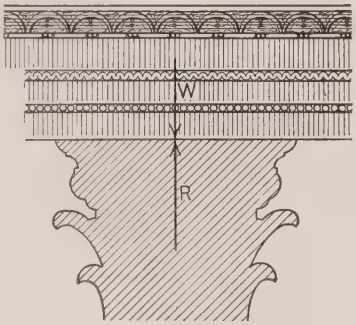


FIG. 4. Action and reaction at the top of a column. $W = R$.

Example 4.

Suppose a boy pulls a sled at a uniform rate over level ground. Friction in this case is exerting an opposite and simultaneous reaction equal to the pull of the boy, as illustrated in Fig. 6.

Example 5.

Suppose the boy pulling a sled, as in Example 4, increases his speed. *Two* reactions now oppose the pull applied to the sled. One, *friction*, opposes the slipping of the sled over the ground; the other, due to *inertia* (see Art. 7) opposes increase of speed. These two together are equal and opposite to the pull exerted on the sled. (See Fig. 7.)

FIG. 5. Action and reaction at the foot of a column. $W' = R'$.

3. The reaction necessary. Suppose we have an ordinary spring balance lying upon a perfectly smooth horizontal table. By " perfectly smooth " table we are to understand one along which the balance can be drawn without friction (a condition which of course

can only be realized approximately for any actual table). If now we take hold of the ring, we may slide the balance steadily along the table, once started, with no sensation of effort; we will be *unable to pull upon it* unless we increase its speed. But suppose we pass the hook of the balance over a pin and then pull. We *may now apply a force* with the hand, the spring will be extended, and the reading of the graduated scale will serve as a measure of this action tending to move the balance. The pin, therefore, provides a condition necessary before force can be applied, which, if we replace the pin by a second balance like the first, we find to consist of *an equal action in the opposite direction.*

FIG. 6. Action and reaction, sled drawn at uniform speed.

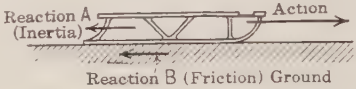


FIG. 7. Action and reaction, sled drawn at increasing speed. Action = Reaction A + Reaction B.

In Example 1 of the preceding article, neither boy can pull on the rope unless the other provides the opposing reaction. The column in Fig. 5 cannot push against the floor unless the floor pushes upward against it. In all instances, therefore, in which we find force exerted two conditions are present:

(*a*) *Two bodies* are always involved. What we term *force* is the "push, pull, attraction, repulsion, rubbing, pressure, etc., of *one body on another.*"

(*b*) When one body, A, exerts force upon a second body, B, body B exerts an equal and opposite action on A, called the *reaction* of B. In speaking of the body A alone, we shall have frequent occasion later to include as among the "forces" applied to it, the reaction of the second body B.

4. The effects of force. A force (i.e., the pressure, attraction, rubbing, etc., of some other body) applied to a given body may produce *one or both of two effects* upon that body:

First, it may *change the state of motion of the body* acted on; that is, may cause the body to move if at rest, or to move faster or slower or in a different direction if already in motion. The state of motion of a body under the action of no force (or of balanced forces) is either absolute rest or uniform motion.

Or, second, the force may *alter the shape of the body.* For exam-

ple, the force may stretch a rope; it may compress a column; it may extend or compress a spring; it may deflect a beam. As all known substances are compressible, and also may be elongated, some change of shape is always produced by an applied force, though perhaps the change may be so slight as to be negligible.

5. Definition of force. From the preceding discussion, we see that *force* is the *name given to the action which one body may exert upon another*, that the effects of such actions may be either change of motion or change of shape, and, finally, that actions between bodies must be mutual or in pairs. Hence we may now state as our definition of force:

A force is one of a pair of equal, opposite, and simultaneous actions between two bodies by which the state of their motion is altered or a change in the form of the bodies themselves is effected.

6. The measurement of force. The *quantity* of either of the effects of force on any given body is found to depend directly upon the quantity of force applied. Hence a force may be measured either:

(*a*) *By the velocity that it will impart to a given body in a given time*, or

(*b*) *By the amount of change of shape which it produces* in a given body.

Method (*a*) enables us to find the force required to start a body from rest, to stop a moving body, or to change the velocity of a body by a stated amount in a given length of time.

Method (*b*) is used in comparing two forces by determining the amount that each will stretch a spring, or the angle through which each will twist a given rod or wire. The spring balance, Fig. 8, is a familiar instrument for measuring forces in this manner. The elongations of the spring, under the action of known forces, are marked on the scale of the instrument.

FIG. 8. Spring balance.

Thus the elongation due to the force with which the earth pulls on a 1-lb weight is marked 1 lb, so that a weight of 1 lb equals a force of 1 lb, and any action which elongates the spring the same distance also equals a force of 1 lb.

The English unit of force is the pound, and is the gravity pull on a mass of 1-lb weight.

7. Positive and passive reactions. In many cases, either one of the mutual actions exerted between two bodies may be regarded as the *positive action* or the *force*, and the other one regarded as the *reaction* resisting the force. For instance, in Example 1, Art. 2, either one of the two boys may be regarded as exerting the force, and the other one as exerting the equal and opposite reaction.

There are, however, three instances in which we cannot assume either action to be the force. These are:

(*a*) *Friction.* In Example 4, Art. 2, it is impossible for us to suppose that the friction is exerting the force and that the boy is exerting the reaction. When friction is one of the equal, opposite, and simultaneous actions, it is necessary to regard the friction as the reaction and the other action as the positive action or force.

(*b*) *Elasticity.* Where the resistance of some body due to its *elasticity* (or resistance to a change of shape) is one of the actions, we cannot regard this as the positive action (unless it is tending to produce motion, as in a spring under tension). In Example 3, for instance, it would not be proper to assume that the table is exerting the force and that gravity is exerting the reaction.

(*c*) *Inertia.* Where the resistance of the body to change of velocity due to its own *inertia* * is one of the actions, we must usually regard this as the reaction (except in certain instances where the velocity of a body is being destroyed, as in cases of impact between bodies).

These three types of reaction can be *developed by* the application of a positive action or force only. We could not logically think of their acting independently to develop an accompanying equal reaction. Nevertheless, where the positive action does exist, as, for illustration, in the gravity pull upon the book, Example 3, we may, when convenient, *think of the reaction of the table as an equal force* upon the book, and may thus speak of the book as under the

* Inertia. Inertia is to be regarded, not as a force, but as a *property of all matter.* "A body at rest will remain at rest, a body in motion will continue to move in the same direction and at the same speed, unless compelled to change its state of rest or motion by the application of an *external force.*" The property of a body because of which force is necessary in order to produce change in its state of rest or motion is spoken of as the *inertia* of the body. Whenever the velocity of a body is changing — increasing, decreasing, or changing direction — therefore, we must consider as *one of the reactions present, the resistance to change of motion* due to the body's inertia. (For further discussion of inertia reactions and force, see Chapter X.)

action of *two forces*, without regard to their source; a downward force equal to the weight of the book, and an equal upward force, the reaction of the table.

This idea of considering a given body apart from its surroundings and acted on by forces only, although purely imaginary, is important in the solution of problems in mechanics. In such cases it is the practice to disregard the sources of the forces acting, and to conceive of a body from which all other bodies that act upon it have been removed and their places taken by the forces which they exerted. In Examples 1 and 2 of Art. 2, considering the rope under the action of two forces, it is immaterial whether the second force is caused by the boy or by the post. A body thus regarded as acted upon by forces only is called a **free body**. Free bodies will be discussed in Art. 43.

It should be noted that, although force actions exist in pairs, the *force is only one of the mutual actions between two bodies.* It must not be interpreted that the force is twice its proper value. If the first boy exerts a pull of 10 lb, the second boy must exert a 10-lb reaction, and the force on the rope will be 10 lb (not 20 lb) transmitted from one boy to the other.

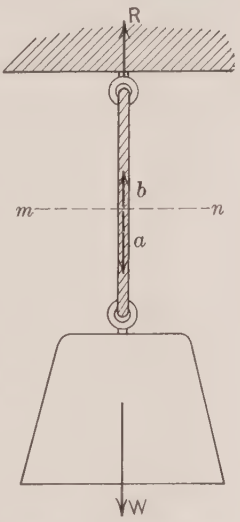


FIG. 9. Tension. a b, both away from any section.

8. Stress. Tension, compression, and shear. The internal force which is caused in a body, due to an external force, is called a **stress.**

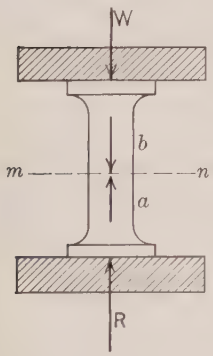


FIG. 10. Compression. $a = b$, both toward any section.

Where the actions are such as to tend to *pull the parts of the body apart,* the body is said to be under **tension.** This is the case, for example, of a rope tied at one end to a rigid support and sustaining a weight as in Fig. 9. Here gravity is exerting a downward pull W on the rope equal to the weight supported, and the support is exerting an equal upward reaction R. These external actions set

up *internal actions* or *stress* in the rope and, at any section of the rope as *m–n*, the actions are opposite and directed *away from* the section as shown. If, however, $W = 200$ lb and we may neglect the weight of the rope below the section, each action is 200 lb, and we say the *tension* is 200 lb, *not* 400 lb.

When the actions are such as to tend to *push the parts* of a body *together*, the body is said to be under **compression.** Illustration of this is the column of Fig. 10, rigidly supported and carrying a load W. At any section of the column, as *m–n*, the actions are opposite and directed *toward* the section. If $W = 1000$ lb and we may neglect the weight of the post above the section, each action is 1000 lb and the compression is 1000 lb, *not* 2000.

Another form of stress, in which the fibres of the material are subjected to a *cutting* action, as in cutting a piece of metal with the tin-shears, is called **shear.**

Stress will be discussed again in Art. 114.

9. Center of gravity. One of the most common forces is that of gravity, that is, the *weight* of a body acting always vertically downward. This force, which is the effect of the gravity pull on all the particles in a body, is, for convenience, usually considered to act at a single point called the **center of gravity.**

The center of gravity of a body is the point at which its weight may be considered to act, no matter in what position the body is placed. It is at the geometrical center of the body, and is the point about which the body will balance. If a body is homogeneous and of regular geometric shape, the position of the center of gravity will be evident from inspection.

Thus the center of gravity of a *uniform timber or bar* is on its central axis at the middle of its length, and its weight can be considered to act at this point, in whatever position the timber or bar may be placed. The center of gravity of bodies of other shapes will be considered in Art. 38.

10. Motion. One of the effects of a force is its tendency to cause motion. **Motion may be defined as change of position.** The change in position of a body is necessarily observed by comparing its successive positions with that of some other body, for the time being regarded as fixed. *All motion, therefore, is relative.*

A body may be in motion with respect to one body and at rest with respect to another. Thus, a man sitting in a railroad train which is not moving (as Fig. 11) is at rest with respect to both the

train and the ground. When the train is moving he is at rest with respect to the car but in motion with respect to the ground. If he walks to the rear of the car at the same rate of motion as the car is

Courtesy The New York, New Haven and Hartford Railroad Company.

FIG. 11. Diesel-powered streamline train.

moving forward, he is at rest with respect to the ground. Or if he moves across the aisle from one side of the car to the other, he has motion in one direction with respect to the car and in a different direction with respect to the ground. His path of motion with respect to the ground will be a diagonal line from one side of the track forward to the other side.

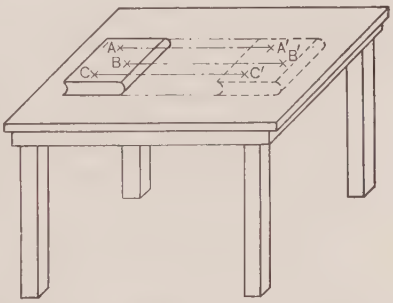


FIG. 12. Translation; all points move the same distance along parallel paths.

When a body moves in such a way that each point in the body passes over a path equal and parallel to the path of every other point, the body is said to have motion of **pure translation.** The book on the table, Fig. 12, if pushed straight along the table to a new position so that $AA' = CC' = BB'$, and these paths are parallel, has

motion of pure translation. When motion of translation is considered it is necessary to know its amount, or magnitude, and its direction.

If, however, the motion is such that the body rotates about a line through two fixed points in the body, the body is said to have motion of **pure rotation**. The line about which the body rotates is called the **axis of rotation**. The pulley and shaft, Fig. 13, when driven by the belt, have such a motion. When motion of rotation is considered it is necessary to know the position of the axis of rotation, and the amount, or measure, of the turning tendency.

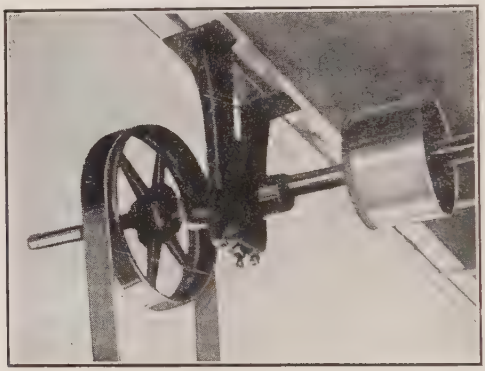


FIG. 13. Rotation; all points move in circular arcs about an axis.

The tendency of a force to cause rotation is called its **moment or torque.** It depends upon two factors, the amount of the force and the distance that it acts from the axis of rotation. The moment of a force is equal to the product of the force multiplied by the perpendicular distance from its line of action to the axis of rotation. The torque on the pulley in Fig. 13 equals the pull on the belt times the radius of the pulley; it would be increased by a greater pull on the belt or by the same pull on a larger pulley.

Frequently a body may have motion of both translation and rotation at the same time, as the wheels of an automobile rotating as the car moves along the street. The wheels have the same motion of translation as the body of the car, and also the motion of rotation about their axles.

The term **displacement** means the distance moved and the direction of the motion, as a displacement of 5 ft up, or of 10 ft

to the right. It does not take into account the time of the motion.

The term **velocity** means the *rate* of motion and does take into account *time*, as well as distance and direction. Velocity is measured by the displacement which occurs in one unit of time, as 30 miles per hour north, or 20 feet per second to the right. Since all motion is relative, velocity, which fixes the rate of motion is also relative and a body may have at the same time different velocities (either in amount or direction) with respect to different bodies to which the motion is referred.

The term **equilibrium** is used to express a condition of rest, or of uniform motion without change, which occurs when all the forces acting on a body are balanced. A body is in equilibrium when there is no tendency toward motion of translation or of rotation.

The principle of moments will be considered in Art. 33 and its applications to levers, parallel force systems, and center of gravity in the succeeding articles of Chapter V. Torque will be considered in Art. 76 and 79 in Chapter X.

PROBLEMS

1–1. A weight is suspended by a cord from a hook screwed into a beam. Point out the action and the reaction:

(*a*) At the point of attachment of the cord to the hook.
(*b*) At the point of attachment of the weight to the cord.
(*c*) Between the hook and the beam.
(*d*) At any transverse section of the cord.

1–2. A man lifts a heavy weight. Point out the action and the reaction:

(*a*) Between the man's hands and the weight.
(*b*) Between the man's feet and the ground.
(*c*) How do the amounts of the action and reaction in (*b*) correspond with those in (*a*)?

1–3. A chandelier weighing 30 lb is supported from the ceiling by a chain 4 ft long weighing 4 lb. What is the amount of the force:

(*a*) Of the chandelier on the chain?
(*b*) Of the chain on the chandelier?
(*c*) Of the chain on the ceiling?
(*d*) Of the ceiling on the chain?

1–4. What is the tension in the chain of Problem 3 if the weight of the chain itself can be considered small enough to be neglected? If the weight of the chain is considered, what is the tension in the chain 2 ft above the chandelier?

1–5. A stone coping block weighing 400 lb rests upon a section of brick wall the weight of which is 3000 lb. The section of the wall rests, in turn, upon a concrete footing the weight of which is 1600 lb. What are the amounts of the actions and reactions: (*a*) At the top of the brick wall? (*b*) At the top of the concrete footing? (*c*) At the bottom of the footing?

1–6. A roof truss weighs 1600 lb and supports 6000 lb of roof load. It is supported by two columns symmetrically placed, each weighing 4000 lb. Find the compression on the footing under each column. What are the amounts of the actions and reactions between roof load and truss, truss and each column, column and footing?

1–7. What actions and reactions are caused by the force of one's foot on the brake pedal of an automobile?

1–8. What actions and reactions cause and oppose motion when an automobile is started from rest? When it is being driven at uniform speed?

1–9. An airplane takes off from a level landing field. What is the action on the plane causing motion? What are the reactions opposing motion? What is the source of the action?

1–10. A heavy timber is being hauled by means of a rope and tractor at uniform speed along a level road.

(*a*) What is the action on the timber keeping it moving? The reaction opposing its motion?

(*b*) Point out the action and reaction at the point at which the rope is attached to the tractor.

(*c*) How does the tension on the rope compare with the friction between the timber and the road?

(*d*) How does the "tractive force" between the wheels of the tractor and the ground correspond with the tension in the rope? Explain.

1–11. Answer questions *a–d* if the timber of Problem 1–10 is being hauled at a constantly increasing speed.

1–12. If the friction between the timber of Problem 1–10 and the road is 600 lb, and the amount of the inertia is 50 lb, find (*a*) the tension in the rope when the tractor is starting off; (*b*) the tension in the rope when the tractor is traveling at uniform speed; (*c*) the action and reaction at the ring where rope is tied to tractor, when starting; (*d*) the action and reaction at the ring when the speed is uniform.

SUMMARY

A **force** is one of a pair of equal, opposite, and simultaneous actions between two bodies by which the state of their motion is altered or a change in the form of the bodies themselves is effected.

One of the pair is called the action, the other the reaction. There are three so-called passive reactions which cannot exist until brought into action by a positive force. These are: (*a*) friction, (*b*) elasticity, (*c*) inertia.

The **effect of a force** may be:

(*a*) To cause a change in the state of motion of a body.
(*b*) To cause a change in the shape of the body.

A force may be measured either:

(*a*) by the velocity that it will impart to a given body in a given time, or
(*b*) by the amount of change of shape which it produces in a given body.

The English **unit of force** is the pound, and is the gravity pull on a mass of 1 lb weight. The elongation of a spring balance due to a 1-lb weight is marked 1 lb, and any action which elongates the spring the same distance is a force of 1 lb.

A free body is one that is regarded as separate from all other bodies and acted upon only by forces. (See also Art. 43.)

A **stress** is an internal force caused in the fibres of a body by an external force called the **load**. If the stress is a pull, it is called **tension**; if it is a push, **compression**; if a sliding, cutting action, **shear.** (See also Chapter XV.)

The **center of gravity** of a body is the point at which its weight may be considered to act, no matter in what position the body is placed.

Motion is change of position, relative to some body considered fixed. A body has motion of **translation** if each point in the body passes over a path equal and parallel to the path of every other point. Straight-line motion is the simplest form of motion of translation. A body has motion of **rotation** if it rotates about an axis through two fixed points in the body.

The **moment** or **torque** of a force is its tendency to cause rotation about an axis. It is equal to the product of the force multiplied by its moment arm: the arm being the perpendicular distance from the line of action of the force to the axis or center of rotation. (See also Art. 33.)

Displacement states both the distance and the direction that a body moves.

Velocity is the rate of motion of a body; it equals the displacement divided by the time. (See also Art. 52.)

Equilibrium expresses a condition of rest, or of uniform motion without change, which occurs when all forces are balanced.

SYMBOLS AND ABBREVIATIONS

P = pull on rope
W = weight or load
R = reaction

lb = pound
ft = foot
in. = inch

CHAPTER II

COMPOSITION AND RESOLUTION OF VELOCITIES AND FORCES

Problems in mechanics can be solved either graphically, by the use of geometry and mechanical drawing, or mathematically, employing arithmetic, algebra, trigonometry, and in some problems not included in this book, the calculus.

The material to be studied in Chapter II describes the procedure for solving certain problems graphically. The meaning of vector quantities and the graphical representation of vector quantities, particularly forces, are explained. The method of the force parallelogram for the composition and resolution of forces, is described. The methods of the force triangle and the force polygon are also considered. These methods will be used in subsequent chapters to analyze force systems in actual machines and structures.

The graphical method also provides a means of correctly visualizing the force diagram in cases where the actual solution is made mathematically. For that reason a thorough understanding of it is important. Evidently, unless the diagram of forces is correctly drawn or sketched, the mathematical solution of that diagram cannot give a correct result, even though otherwise the mathematical computations are correct.

11. Vectors. Representation of velocities and forces by lines. Velocity, the rate of motion, is described by stating its amount (or magnitude), its direction, and the point from which it starts. Similar attributes also describe a force. Such quantities are called **vector quantities.**

A vector quantity can be represented by a straight line drawn to or from the point of action, of a definite length to some convenient scale, in the proper direction and with an arrow head to indicate the direction. Such a line, the length of which determines the magnitude of a quantity, and the direction of which determines the direction in which that quantity acts, is called a **vector.**

Thus the forces R and W illustrated in Fig. 9 would be represented graphically by letting some point O be the point at which

the forces act or start, as at the center of the ring on the weight, and drawing from that point two lines, one vertically down and one of equal length vertically up with an arrow head at the end of each. If the weight were 15 lb, and a scale of 1 in. = 10 lb were used, then each line would be $1\frac{1}{2}$ in. long.

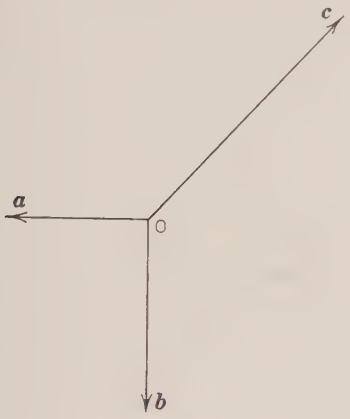


FIG. 14.

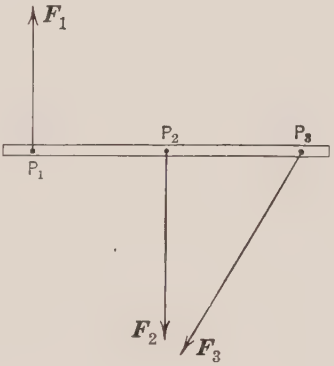


FIG. 15.

The lines **a**, **b**, and **c**, in Fig. 14 represent velocities of 15 fps west, 20 fps south, and 30 fps northeast, respectively, from the point O, on a scale of $\frac{1}{4}$ in. = 5 fps; i.e., line **a** is $\frac{3}{4}$ in. long, line **b** 1 in., and line **c** $1\frac{1}{2}$ in.

Lines $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, Fig. 15, represent forces of 6, 8, and 10 lb, respectively, applied to a rod at the points P_1, P_2, and P_3, and acting in the directions indicated, the scale being $\frac{1}{8}$ in. = 1 lb (i.e., $\mathbf{F}_1$ $\frac{3}{4}$ in. long, $\mathbf{F}_2$ 1 in., $\mathbf{F}_3$ $1\frac{1}{4}$ in.).

PROBLEMS

Note: In all graphical work draw accurately, with fine lines, to as large a scale as the page permits without inconvenient ratios. State the scale used, and place on all lines their values in velocity or force units.

2–1. Represent by lines starting from a common point, and showing direction and amount, velocities of 50 fps south; 30 fps west; 20 mph southeast.

2–2. Represent by vectors the forces on a small ring by which 27 lb is suspended from a spring balance.

2–3. Represent by vectors forces of 10, 15, 18, and 22 lb acting at a point, the angle between the first and second being 30°, that between the second and third 45°, and that between the third and fourth 60°.

2–4. From a straight rod AB, 7 ft long, the following forces act upward: 13 lb at a distance of $1\frac{1}{2}$ ft from A; 38 lb $1\frac{1}{2}$ ft from B. Also the following forces act downward: 5 lb $\frac{1}{2}$ ft from A; 10 lb $2\frac{1}{2}$ ft from A; 1 lb at the center of the rod; 15 lb $2\frac{1}{2}$ ft from B; 20 lb $\frac{1}{2}$ ft from B. Represent this force system by a diagram drawn to scale, stating the scales used. (Use separate scales for the *apparatus* scale and the *force* scale.)

2–5. Represent graphically three forces at a point O, one of which is 18 lb down in a vertical line, and two upward at 45° to right and left of $12\frac{1}{2}$ lb each.

2–6. At a point O, $18\frac{1}{2}$ lb acts vertically down, $6\frac{1}{2}$ lb vertically up, 12 lb at 30° above the horizontal to the right, and 12 lb at 30° above the horizontal to the left. Draw vectors.

12. Composition of velocities. In a preceding article it was pointed out that a body might have one velocity with reference to a second body and the second body might have some other velocity with reference to a third. In such cases it is often important to determine what the velocity of the first body is with reference to the third. This velocity can be determined by properly combining the two velocities. The velocity obtained in this way is called the **resultant velocity,** and the process by which it is obtained is called the **composition of velocities.** The two velocities from which the resultant is obtained are called the **component velocities.** The resultant velocity shows the direction and rate of motion of a body to which two velocities different in amount or direction or both are simultaneously imparted.

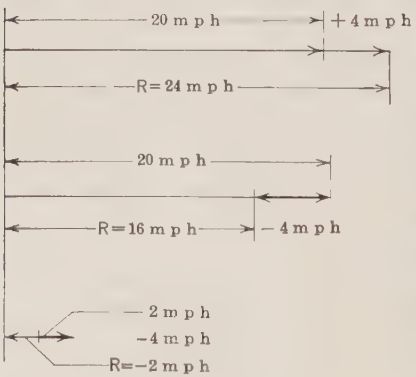


FIG. 16.

If the component velocities are along *the same straight line* and in the *same direction*, it is evident from a little consideration that their resultant equals their sum. If the two velocities are in *opposite directions* their resultant equals their difference. Thus, if a man walks at the rate of 4 miles per hour forward along the aisle of a car going 20 miles per hour, his velocity with respect to the ground will be: $20 + 4 = 24$ mph. If he walks toward the rear of the same car, his velocity will be: $20 - 4 = 16$ mph. If the velocity of the car were 2 mph instead of 20, and the man walked toward the rear at

4 mph, his velocity with respect to the ground would be: $2 - 4 = -2$ mph, that is, 2 mph in the direction opposite to that of the car.

These three cases are represented graphically as in Fig. 16. The scale before reduction was $\frac{1}{8}$ in. = 1 mph.

For this case the mathematical solution is probably simpler and quicker than the graphical, and will be considered in Art. 27.

13. Parallelogram of velocities. In the preceding article the component velocities were along the same straight line. If they are along lines making *an angle* with each other, their resultant can be found by the use of the parallelogram law.

In Fig. 17 imagine the point O to be moving with a velocity Y along the blade of the T-square AB. If the T-square is at rest, in one unit of time the point will move along the path OP to P. If, on the contrary, point O is at rest on the blade, but the whole T-square slides along the drawing board to a new position CD, with a velocity X, in one unit of time O will pass along the path OQ to Q. Finally, if point O moves along the blade with a velocity Y *and at the same time* the T-square slides along the drawing board with the velocity X, in one unit of time the point O will be carried along the board a distance OQ and across the board a distance OP, or it will be at R, which is the further extremity of the diagonal of the parallelogram $OPRQ$ shown dotted in Fig. 17.

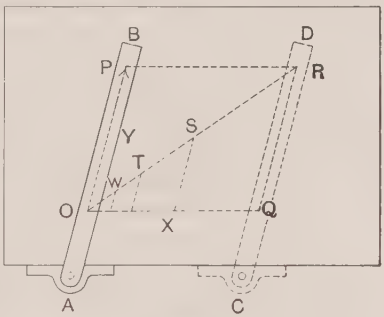


Fig. 17.

If a unit of time half as big had been chosen, point O would have moved half as far along the blade, and the T-square would have carried it half as far along the board. The point O, therefore, in half the time would have moved to S. If the unit of time is still further subdivided, other points, as T, W, etc., may be located. *All these points will lie on the diagonal OQ of the parallelogram.* Hence the path traveled by the point O to which the two velocities X and Y are imparted is along the diagonal of the parallelogram; and, since the position R is reached in unit time,

the length of this diagonal, OR, represents correctly the resultant velocity of the point.

Therefore, **to find the resultant velocity** of a body to which two component velocities not in the same straight line are imparted, represent the two component velocities in their proper directions and to a convenient scale, as the adjacent sides of a parallelogram. Complete the parallelogram, and, from the point of intersection of the sides representing the component velocities, draw the diagonal. This **diagonal represents the resultant velocity** both in direction and in amount, in accordance with the scale assumed.

14. Parallelogram of forces. The principles just stated for velocities apply to the composition of any vector quantities. The case of chief importance in mechanics is the **composition of forces.** When two or more forces act at a single point* on a body, the single force which will produce the same effect as the several forces is called the **resultant force.** Any one of the original forces is called a **component force.**

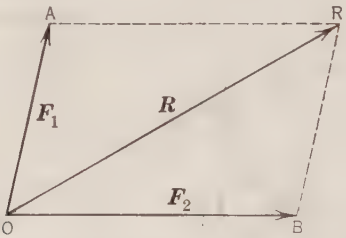


FIG. 18. Diagonal R is the resultant of components $\mathbf{F}_1$ and $\mathbf{F}_2$

The parallelogram law may be stated, for forces thus:

If two forces acting at a single point on a body be represented in magnitude and in direction by two straight lines drawn from that point, and if a parallelogram be constructed on these lines as adjacent sides, the *resultant* of the two forces will be represented in both magnitude and direction by the *diagonal* of this parallelogram, drawn from the point. (See Fig. 18.)

This proposition may be regarded as a corollary of the parallelogram of velocities, since forces may be measured by the velocities which they will impart in a given time (Art. 6). Thus in Fig. 18 let the vectors OA and OB

* The point of application of a force may be considered as being *at any point along the line of action of the force.*

represent two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting at a point O; then according to *some* scale each vector will *also* represent the velocity the respective force would impart in one unit of time. The resultant of these two velocities is represented by the diagonal OR of the parallelogram, that is, the two forces acting together will impart a velocity equal to OR. But a force equal to OR would also impart this same velocity, and therefore the diagonal of the parallelogram represents the force which will produce the same effect as the two original forces and consequently is their resultant.

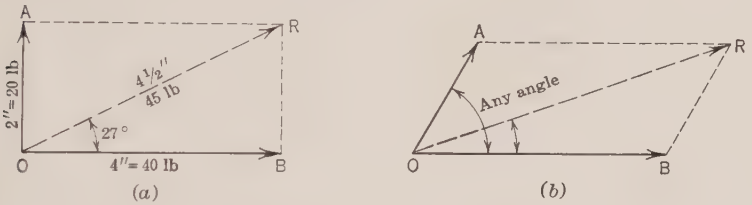


FIG. 19. Diagonal OR is resultant of forces OA and OB.

Example.

Given a vertical, upward force of 20 lb, and one horizontal, toward the right, of 40 lb; determine their resultant.

Solution. From a point O draw a vertical line OA to scale, Fig. 19. Let the scale be 1 in. = 10 lb; then OA will be 2 in. long. Draw a horizontal line OB 4 in. long. Complete the parallelogram by drawing BR parallel to OA, and AR parallel to OB. Draw the diagonal from the point O, i.e., OR, which represents the resultant of OA and OB. OR scales $4\frac{1}{2}$ in., and angle BOR measures 27° on a protractor, therefore the resultant of the two given forces is found to be:

45 lb acting 27° from the horizontal. *Ans.*

The angle between the component forces in the foregoing example was 90°, but the method may be used equally well at whatever angle the forces act.

While this is essentially a graphical method, it can be adapted to a mathematical solution, as described in Art. 28.

The parallelogram is most easily constructed by the method of mechanical drawing, with T-square and triangle or with two triangles. Having drawn the two given lines to proper scale and in proper direction, lay one edge of triangle along one of the lines, hold the T-square or other triangle against another edge, and slide the first triangle along into the position wanted; its edge will remain parallel to its first position. (See Fig. 20.) Follow a similar procedure with the second line.

Lacking triangles, if compasses or a protractor are available, the geometrical construction may be used, but the geometrical construction is not usually so convenient.

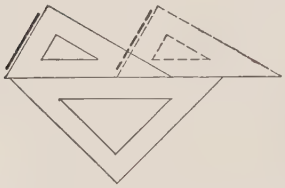


FIG. 20. A method of drawing parallel lines.

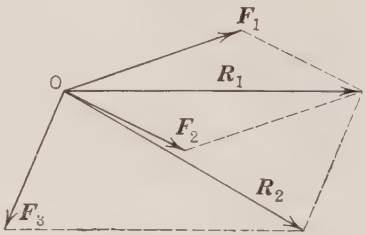


FIG. 21. $\mathbf{R}_2$ is the resultant of forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$.

15. Composition of more than two forces. Any number of forces acting at a single point and lying in the same plane may be combined by repeated applications of the parallelogram law. In this method, the resultant of two of the forces is first found, then the resultant of this first resultant and the third force is found, and so on, until all the forces have been combined.

Thus in Fig. 21, the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ may be combined to find the resultant $\mathbf{R}_1$ of the two. The resultant $\mathbf{R}_1$ and the third force $\mathbf{F}_3$ may be combined to find the resultant $\mathbf{R}_2$. Force $\mathbf{R}_1$ is equivalent to the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$; therefore the force $\mathbf{R}_2$ is equivalent to the three forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ and is their resultant.*

An optional method, which in some cases is more direct than the preceding one, is that of the force polygon. The force polygon and the force triangle are developed in Art. 19.

PROBLEMS

2–7. Given a southerly force of 32 lb and one easterly of 44 lb, draw the force parallelogram, and determine the resultant in amount and direction.

2–8. Draw an angle of 75° and upon its side lay off vectors of 5 and 9 units respectively. Find their resultant, in magnitude and direction.

2–9. Forces of 6 and 15 lb act from a point at an angle of 60°. Find the resultant and the angle between the resultant and each force.

2–10. Find the resultant of a horizontal force of 180 lb and one of 90 lb at an angle of 60° to the horizontal.

* It should be noted that this method is essentially graphical, and any attempt to use it in the mathematical solution of later articles will lead to difficulty with angle determination.

2–11. Given 50 lb acting upward and at an angle of 40° to the right of the vertical, and 30 lb upward and 20° to the left of the vertical, find their resultant in amount and direction.

2–12. A force of 125 lb acts at an angle of 50° to a force of 100 lb. Find the resultant.

2–13. A man is rowing with a velocity of 5 mph at right angles to a current which has a velocity of $2\frac{1}{4}$ mph parallel with the shore. Find his velocity and direction with reference to the shore.

2–14. A canal boat is hauled by mules on both banks. The rope to one bank makes an angle of 20° with the bank, and applies a pull of 300 lb. The second rope makes an angle of 15° with the other bank and applies a pull of 140 lb. Assuming the canal banks to be parallel, find the amount of the resultant pull and its direction with respect to the bank.

2–15. A train is moving at the rate of 35 mph in a northerly direction. The wind is blowing at the rate of 20 mph from the southwest. Find graphically the direction in which a weather vane would point if it were attached to the top of the train.

2–16. In Example 2–15, suppose the wind is blowing with a velocity of 40 mph from the southwest. How fast must the train move in order that the weather vane shall point towards the northwest?

2–17. If an airplane is flying on a north bearing, with a speed relative to the air of 100 mph, draw vectors to determine its speed and direction relative to the ground when there is a wind

(*a*) from the south with a 25-mph velocity.
(*b*) from the north with a 25-mph velocity.
(*c*) from the east with a 30-mph velocity.
(*d*) from the southeast with a 50-mph velocity.

2–18. Three posts are placed in the ground so as to form an equilateral triangle. An elastic cord is stretched around them, the tension in which is 34 lb. Find the resultant force acting on each post and its direction.

2–19. Forces of 6 and 10 lb act at right angles to each other. Find the amount and direction of their resultant.

2–20. Find the resultant of the following three forces acting at a point *O*: 200 lb vertically downward, 140 lb horizontally toward the right, and 160 lb downward to the right at an angle of 30° to the horizontal.

2–21. Find the amount and direction of the resultant of the combination of forces shown in Fig. 53(*a*).

2–22. Find the resultant of two forces of 100 lb each, when the angle between them is: (*a*) 0°, (*b*) 30°, (*c*) 60°, (*d*) 90°, (*e*) 120°.

16. The resolution of forces. Any quantity having both amount and direction, as a given force or velocity, may be resolved into *two or more components* which will produce the same effects as the given quantity. This process is known as the **resolution of forces** (or motions, etc.).

Thus, in Fig. 22, if OR is a given force, the component forces represented by the adjacent sides of any of the parallelograms which may be formed on OR as a diagonal, such as OA and OB, or OX and OY, will have OR for their resultant and will therefore be in every way equivalent to OR, and may be at any time substituted for it. As an indefinite number of parallelograms may be drawn with OR as the diagonal, there will be an indefinite number of sets of components into which every force may be resolved. It is evident, therefore, that in order to resolve a force into definite components, enough data must be given to determine the form of the parallelogram.

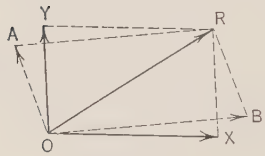


Fig. 22.

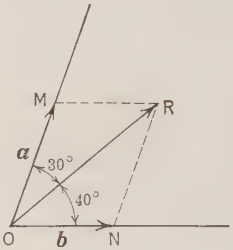


Fig. 23.

Thus if components **a** and **b**, Fig. 23, are to make known angles with **R**, as, say, 30° and 40°, draw **R** to proper scale, then from O draw lines of indefinite length, in the proper directions for **a** and **b**. Then, from the extremity of **R**, draw RN and RM parallel respectively to **a** and **b**. These will cut off lengths from O, determining the values of the components **a** and **b.**

If **R** is to be resolved into forces **a** and **b** and either **a** or **b** is known both in amount and direction, points R and either M or N are fixed, from which the parallelogram may be completed and both **a** and **b** determined.

If components **a** and **b** are known in amount but not in direction, first draw **R** to scale. Then from O, with a radius equal to b, describe the arc op as in Fig. 24 and from R with a radius equal to a describe the arc st. Their intersection fixes the point N. Locate the point M in a similar way, and complete the parallelogram. Then measure the angles between **a** and **b**, or of either with **R**.

17. Resolution into rectangular components. The most important case of the resolution of forces, velocities, etc., is that in

which the given quantity is resolved into two components which are at right angles to one another. Such components are called *rectangular components*. For example, we frequently desire to know the **horizontal component** and the **vertical component** of a given force. Suppose OR, Fig. 25, to be the force. Construct the parallelogram $OXRY$ so that OY will be vertical and OX will be horizontal; then the horizontal component of OR will be the vertical projection of OR, or OX, and the vertical component of OR will be the horizontal projection of OR, or OY.

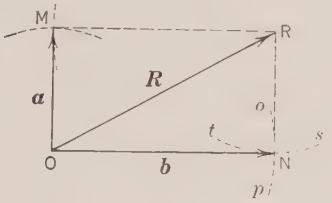


Fig. 24.

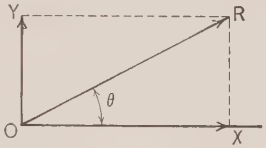


Fig. 25.

It will be seen from the figure that component OX gives *all the effect of OR in the horizontal direction*, and that component OY gives the *total vertical effect*. These components are frequently called the X and Y, or more commonly in practice, the H and V components.

Example.

Given a force of 45 lb. acting at 27° from the horizontal, upward and toward the right; resolve it into its H and V components.

Solution. Take a point O and through it draw a horizontal guide line toward the right. Lay off 27° from the horizontal with the protractor and draw a line OR, to scale, to represent the given force of 45 lb. Suppose a scale of 1 in. = 10 lb is to be used; then OR will be $4\frac{1}{2}$ in. long. Through the point O draw horizontal and vertical lines of convenient length, and from R draw lines to complete the parallelogram on OR as a diagonal, determining the points X and Y. Then scale OX = 4 in. = 40 lb for the horizontal component; and OY = 2 in. = 20 lb for the vertical component.

The mathematical solution for rectangular components is in Art. 29.

18. Use of squared paper in resolving a force into rectangular components. Where a given force is to be resolved into rectangular components graphically, a convenient method is to draw

the force parallelogram on squared paper; the components may then be read directly from the ruled paper.

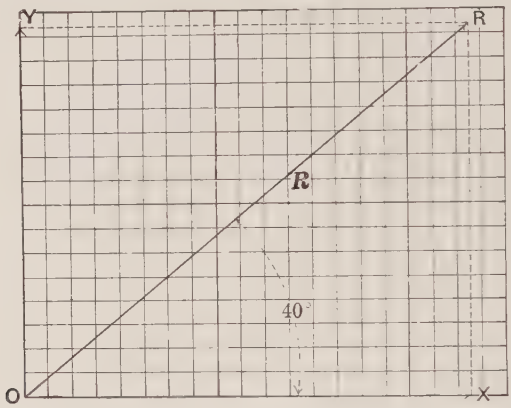


Fig. 26. Rectangular components by the use of squared paper.

Example.

In Fig. 26, **R** is a force of 120 lb, acting at an angle of 40° to the horizontal; to find its horizontal and vertical components.

Solution. Suppose the paper is ruled in $\frac{1}{8}$-in. divisions: a convenient scale is then 1 division = 5 lb. Drawing a line OR, 40° to the horizontal and 3 in. long, to represent the force, and, reading from the paper the value of the components, we have

Horizontal component OX = 18.5 divisions = 92.5 lb.
Vertical component OY = 15.4 divisions = 77.0 lb.

This method makes it unnecessary actually to measure OX and OY to determine their length, or to use a protractor in order to get the components at right angles.

PROBLEMS

2–23. A force of 60 lb has two components, a and b, making angles of 30° and 45°, respectively, with the force. Find the values of the components.

2–24. Resolve a force of 120 lb into two components, one of which shall be a force of 72 lb acting at an angle of 25° with the 120-lb force.

2–25. Resolve a force of 49 lb into two components of 28 lb and 35 lb, and determine the angle between the 49-lb force and each component.

2–26. Resolve a force of 2000 lb acting upward and to the left at an angle of 30° with the horizontal into its H and V components.

2–27. A force of 45 lb acts in a direction inclined 20° to the horizontal. Find the values of its H and V components.

2–28. Resolve a vertical force of 600 lb into components at angles of 50° and 60° with the force.

2–29. Resolve a vertical force of 40 lb into components, one of which is horizontal and the other at an angle of 45°.

2–30. A canal boat is towed by a pull of 180 lb at an angle of 10° with the axis of the boat. Find, using squared paper:

(*a*) The force urging the boat directly ahead in the line of its axis, and

(*b*) The force urging the boat toward shore.

2–31. A body weighing 36 lb rests upon an inclined plane which has an angle of 26° with the horizontal. Find the force perpendicular to the plane, and also the force parallel to it.

2–32. An airship traveled a distance of 120 miles in an hour, in a direction 20° east of north (i.e., N 20° E). If the bearing of the ship was due north and the wind was steady from the west throughout, find the velocity of each.

19. The triangle and polygon of forces. The force triangle is a convenient and somewhat shorter method than the force parallelogram for the composition and resolution of forces. Reference to Fig. 18 shows that, since BR is equal and parallel to OA (which represents the force $\mathbf{F}_1$), the two forces and their resultant can be drawn so as to form a triangle OBR (one-half the parallelogram). The triangle, therefore, is all that *need* be drawn, except that in some cases the student will be able *better to visualize the conditions by constructing the whole parallelogram.*

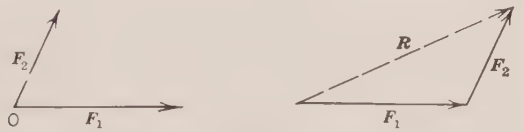


FIG. 27. The force triangle.

Thus, to determine the *resultant* $\mathbf{R}$ of forces $\mathbf{F}_1$ and $\mathbf{F}_2$ in Fig. 27, draw $\mathbf{F}_1$ to scale in the given direction; from its forward end draw $\mathbf{F}_2$ to scale in its given direction; then the vector drawn from the origin of $\mathbf{F}_1$ to the forward end of $\mathbf{F}_2$ is the resultant $\mathbf{R}$.

To resolve a force, as $\mathbf{R}$ in the same figure, into its *components*, draw the force to scale in the given direction; from its origin draw a line in the given direction of $\mathbf{F}_1$; from its forward point draw a second line in the *opposite* direction to force $\mathbf{F}_2$. The intersection

of these two lines completes the force triangle and determines the component forces.

The force polygon is an extension of the principle of the force triangle. The triangle is the special case of a polygon of three sides. Reference to Fig. 28 shows that $\mathbf{R}_1$ is the resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$, and that $\mathbf{R}_2$ is the resultant of $\mathbf{R}_1$ and $\mathbf{F}_3$, and therefore $\mathbf{R}_2$ is the resultant of $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$. Observe that $\mathbf{R}_2$ can be determined directly without actually drawing $\mathbf{R}_1$.

FIG. 28. The force polygon.

Therefore **if two or more forces acting at a point be drawn as vectors, each starting from the forward end of the preceding one, the force vector, drawn from the origin of the first to the forward end of the last force, will equal their resultant in amount and direction.**

PROBLEMS

2–33. Find the resultant of a horizontal force of 270 lb and a vertical force of 90 lb.

2–34. Find the resultant of 12 lb and 16 lb acting at an angle of 53°.

2–35. Find the H and V components of 130 lb acting at an angle of 23° with the horizontal.

2–36. A force of 1000 lb is acting vertically downward from a point O. Find its components acting horizontally to the left and downward toward the right at an angle of 30° with the horizontal.

2–37. Find the resultant of the following forces at a point O: 50 lb vertically upward, 60 lb at an angle of 30° to the right of the vertical, 70 lb horizontally toward the right.

2–38. Find the resultant of 50 lb, 40 lb, and 70 lb, if the angle between the first and second force is 20° and between the second and third is 10°.

2–39. An airplane flies from one airport to another due east of and 200 miles away from the first in 2 hr. If the wind velocity is 40 mph (average) from the south, what is the plane's airspeed and direction of flight, or bearing?

2–40. A plane flying between two airports followed a path 35° west of south (i.e., S 35° W) at an average ground speed of 110 mph. Find the airspeed and direction of flight if the plane encountered an east wind of 25 mph average velocity.

SUMMARY

A vector quantity is one which can be described by stating its point of action (or beginning), its direction, and its amount or magnitude. Motion, displacement, velocity, force, alternating current voltage, etc., are vector quantities.

A **vector** is a line used to represent a vector quantity. It has a definite starting point, a definite length (to scale), and a definite direction (shown by an arrow head).

Composition of forces (or vectors) is the graphical determination of the resultant of two or more forces (or vectors).

Resolution of forces (or vectors) is the graphical determination of the components of a given force (or vector).

A **resultant** of two or more forces is the force which can replace the component forces and cause the same effect as the original forces. If the forces act in the same straight line the resultant will be their sum or difference, depending upon whether they are in the same or opposite directions. If the forces act at an angle with each other, the resultant can be found by the parallelogram law.

The **parallelogram law** is: If two forces acting at a single point on a body be represented in magnitude and in direction by two straight lines drawn from that point, and if a parallelogram be constructed on these lines as adjacent sides, the **resultant** of the two forces will be represented in both magnitude and direction by the **diagonal** of this parallelogram, drawn from the point.

The resultant of more than two forces can be found by use of the parallelogram law, by finding first the resultant of two of the forces, then the resultant of this first resultant and the third force, and so on. Or the method of the force polygon may be used.

The **components** of a force may be found by use of the parallelogram law. The force is drawn as the diagonal of a parallelogram, the sides of which are constructed from the given data regarding the components. (See Art. 16.) Or the method of the force triangle may be used.

The **polygon law** includes the triangle as a special case (a polygon of three sides) and is stated thus: If two or more forces acting at a point be drawn as vectors, each starting from the forward end of the preceding one, the force vector, drawn from the origin of the first to the forward end of the last force, will equal their resultant in amount and direction.

SYMBOLS AND ABBREVIATIONS

F, F_1, F_2, etc. = a force; several forces.
H and V = horizontal and vertical components.
O = point of action of forces; origin of force parallelogram, or of coordinate axes.

OR = diagonal of parallelogram.
X and Y = coordinate axes; velocities; components along OX and OY.
W = weight, or load.
θ(theta) = a given angle.

$\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, **a**, **b**, **c**, **R** = vector forces and resultant. (Symbols for vectors are printed in bold face type.)

lb = pound or pounds.
ft = foot or feet.
in. = inch or inches.
min = minutes.
sec = seconds.
hr = hour.
mph = miles per hour.
fps = feet per second.
fpm = feet per minute.

CHAPTER III

GRAPHICAL SOLUTION OF CONCURRENT FORCES IN EQUILIBRIUM

This chapter treats of equilibrium, and of the graphical solution of systems of concurrent forces in equilibrium in one plane. It makes use of the composition and resolution of forces studied in Chapter II. The mathematical solution will be considered in the following chapter.

The various force systems in one plane are divided into three groups for convenience in studying the subject, and each group is described and analyzed in one of the following chapters. **Concurrent forces** are those whose lines of action meet at a common point. **Parallel forces** are those whose lines of action are all parallel. **Non-concurrent non-parallel forces** are those whose lines of action do not all meet at a common point, neither are all parallel.

20. Equilibrium. **When a body is at rest, or moving uniformly without change in a straight line or in the arc of a circle, it is said to be in equilibrium.** Since one effect of a force is to tend to cause a change in the state of motion of a body, it is evident that a body will be in equilibrium only either when there is no force acting upon it, or else when there are two (or more) forces, so balanced that they do not have the effect of motion. In other words the forces must be such that their *resultant on the body is equal to zero.* Such a balanced force system **is said to be in equilibrium.**

A book lying upon a table is in equilibrium because the force of gravity on the book is balanced by the reaction of the table on the book. An elevator moving up or down with uniform velocity, and a shaft rotating at uniform speed, are also examples of bodies acted upon by balanced forces and therefore of bodies in equilibrium.

If the upward force on the elevator exceeds the forces downward, the elevator will move upward with increasing speed. If the driving effect on the shaft exceeds the retarding, the shaft speeds

up; if the reverse is true, it slows down. In either case the elevator and shaft are *not* in equilibrium.

21. The two conditions of equilibrium. Motion may be either or both of two kinds, motion of translation and of rotation. In order, therefore, that a given system of forces may be in equilibrium, it is necessary that:

1. **All the forces acting together shall have no tendency to produce motion of translation.**

2. **All the forces acting together shall have no tendency to produce rotation about any chosen axis.**

These *two important conditions* will be applied throughout the following chapters in the solution of various force systems.

22. Concurrent forces. Forces whose lines of action pass through a common point are called *concurrent forces.* Thus the lines of action of the forces F_1, F_2, and F_3 of Fig. 28 all pass through the point O; hence these forces are concurrent. The application of the first condition of equilibrium in the preceding article is sufficient for the solution of problems involving forces which are concurrent.

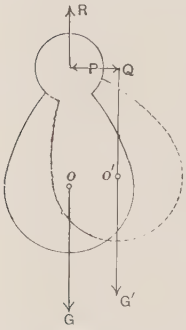


FIG. 29. A two-force piece. For equilibrium the center of gravity must be vertically under the pin at P.

It is obvious from preceding articles that the least number of forces that can produce equilibrium is two. It is obvious also that **a body acted upon by two forces can be in equilibrium only when the forces are equal in amount and act in opposite directions in the same straight line.** In which case the resultant equals zero.

Thus the suspended body shown in Fig. 29, if acted on by *two forces only* (i.e., no friction at the pin P), could be in equilibrium only when the pull of gravity G is in the same vertical line with the reaction R of the pin. In any other position G would have a moment about the pin P which would cause rotation with increasing speed.

The rope, shown in Fig. 9, and the column of Fig. 10, are illustrations of typical "two-force pieces." Such members are always either simple tension or compression members, with the action line of the forces along the axis of the piece.

23. Three concurrent forces in equilibrium. Let **a** and **b**, Fig. 30, be two forces whose action lines intersect at O. Com-

bining these by the parallelogram law, we find their resultant **R**. If a third force is to act with **a** and **b** in such a way that the three shall balance, or in other words, so that the combined effects of **a**, **b**, *and* **c** *shall be zero*, it is evident that **c** *must be equal and opposite to the resultant* **R** and that it must act in the same straight line as **R**.

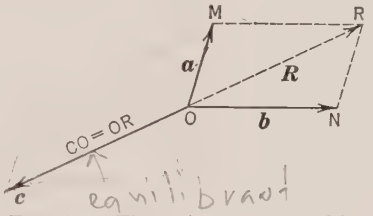


Fig. 30. Three forces in equilibrium.

A force, equal and opposite to the resultant, is called the equilibrant.

Three forces acting at a point and in the same plane will produce equilibrium when the diagonal of the force parallelogram formed on any two forces as sides, drawn from the point of concurrence, is equal and opposite to the third force.

Experimental test of parallelogram law. The truth of the parallelogram law may easily be tested experimentally as shown in Fig. 31. A ring O is supported by two cords OA and OB, in which spring balances, F_1 and F_2 are inserted, and from the ring is suspended a weight, W. Hold a drawing board covered with paper behind the apparatus and indicate the center of the ring O by a pencil dot. Place dots also back of the centers of the cords OA, OB, and OW. Connect these dots by lines showing the directions of the forces, and record beside each line the value of the force in that direction as determined from the balances F_1 and F_2 and the weight, W. Then, on this *direction diagram*, to some convenient force scale, plot F_1 along line OA and likewise F_2 along OB. Complete the *force parallelogram* and scale its diagonal, which should equal W in amount and be in the vertical line OW, Fig. 32.

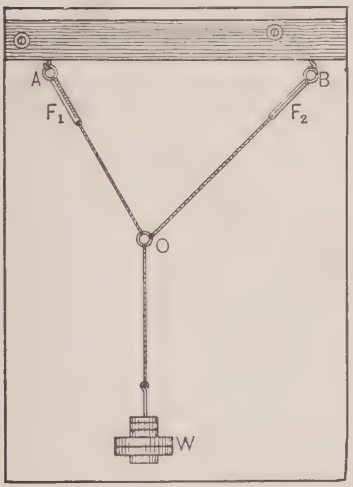


Fig. 31. Apparatus for an experimental study of the parallelogram law.

The data for plotting the direction diagram could also be obtained by measuring the angles with a large protractor. The

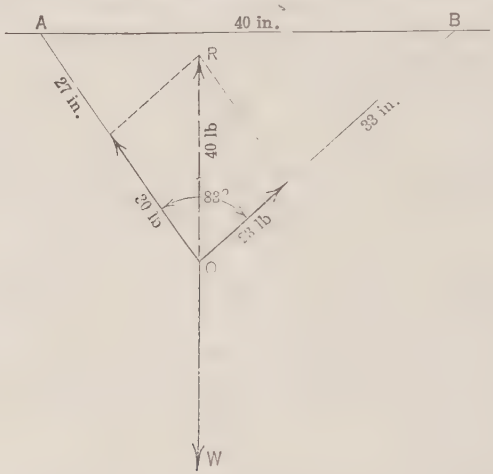


Fig. 32. Direction and force diagram for three forces at the point O.

usual (and more accurate) method on actual structures and machines is by the measurement of the members, in this case

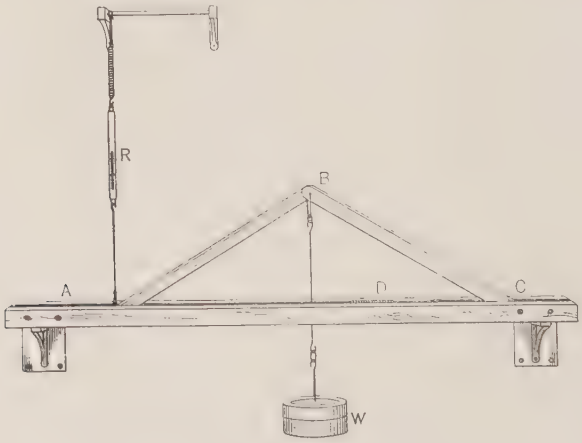


Fig. 33. Simple truss for laboratory study.

the sides of the triangle AB, OA, and OB and noting that AB is horizontal and OW vertical. Note that weight (the gravity

pull) *always* acts vertically. This procedure is described in the Appendix, Art. 5.

A typical case of three concurrent forces in equilibrium is the *simple roof or "A" truss*, Fig. 33. Members AB and CB are in compression and support the load W at B. Three forces act on the pin or joint at B. The force parallelogram is suggested in Fig. 34. Since W must be equal and opposite to the resultant of the thrusts in AB and CB, $\mathbf{R}$ can be scaled equal and opposite to W and resolved into its two components, $\mathbf{F}_1$ and $\mathbf{F}_2$.

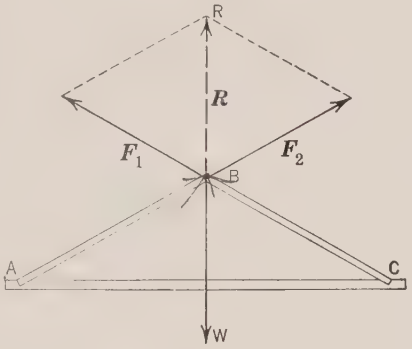


FIG. 34. Force diagram for simple truss. The push of each rafter at B is drawn as a pull in the vector diagram.

These component forces are carried through the members to

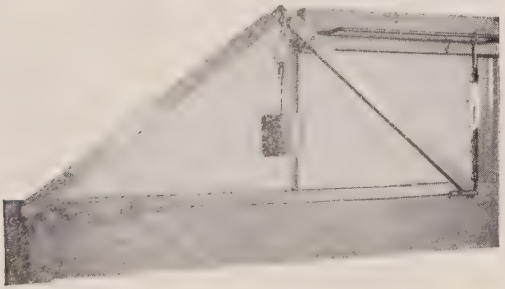


FIG. 35(*a*). Model roof truss.

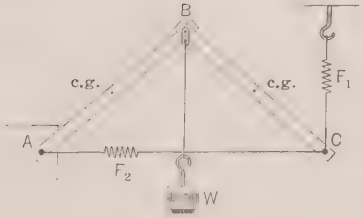


FIG. 35(*b*). Diagram of model roof truss.

the joints A and C, where each can be resolved into its H and V components. Figures 35(*a*) and 35(*b*) show a model truss in which

the spring balances furnish an experimental check of the H and V components.

Another case is the laboratory apparatus of a *roller on an inclined plane*, Fig. 36. This apparatus represents the common case of a body supported on, or being drawn at uniform speed up an incline.

The construction is here such that the friction of the roller on the plane is negligible, hence the reaction $\mathbf{F}_1$ of the plane is perpendicular to the surface. Three forces, the pull $\mathbf{F}_2$, $\mathbf{F}_1$, and the weight W act upon the roller, holding it in equilibrium. The

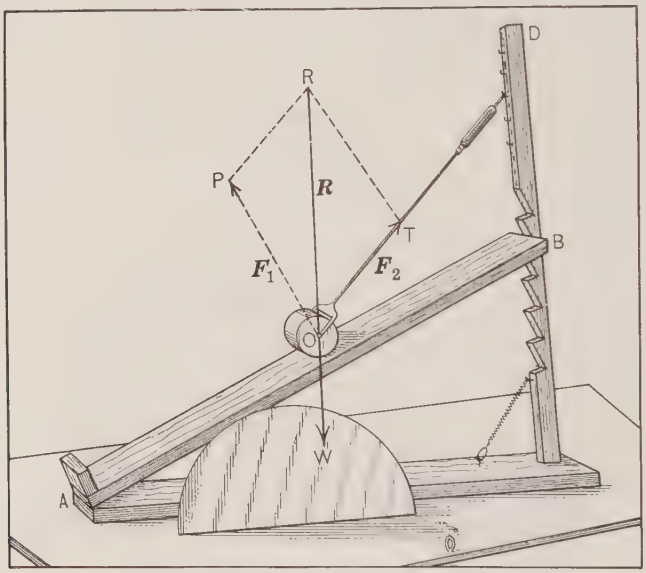


Fig. 36. Inclined plane and roller.

directions of the three forces are all known for any given setting of the plane. Since W must be equal and opposite to the resultant of forces $\mathbf{F}_1$ and $\mathbf{F}_2$, if a force OR be drawn to scale to represent W in amount, the force parallelogram may be completed, having OT and OP as adjacent sides. These lines may then be measured and the values of $\mathbf{F}_1$ and $\mathbf{F}_2$ determined. $\mathbf{F}_2$ may then be checked with the balance reading in the cord, and $\mathbf{F}_1$ by pulling with a balance in the direction OP until the roller is *just free* from the plane.

See Art. 31 for mathematical solution of concurrent forces in equilibrium.

Example.

Figure 37 shows a laboratory model of a simple truss construction in which the horizontal member AB, hinged or pinned at A, and bearing a load W at B, is supported by a tie CB. Suppose $AB = 24$ in., $AC = 24$ in., and $W = 35$ lb. Required: (a) the tension in the tie CB; (b) the thrust of AB against the wall.

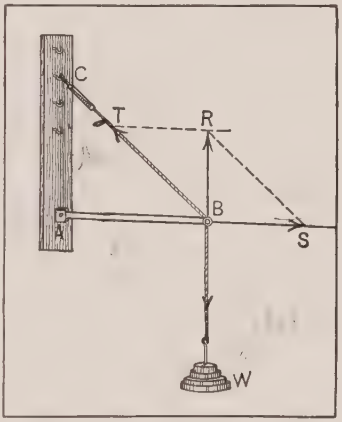


FIG. 37. Stick and tie arranged as a simple truss.

Solution. It is obvious that the ring B is here in equilibrium under the action of three concurrent forces, viz.: the vertical downward pull of 35 lb; the pull in the tie which must be in the direction BC (for if the tie were cut it is evident that B would move downward); and the reaction of the stick AB which is along AB and evidently in the direction BS (for if AB were cut, the ring B would fall inward toward the wall).

The student should notice that in this example, as also in the previous illustration of the roof truss, when the force is a *push* as in the stick AB, or in the rafters of the truss, the force vector is drawn as a *pull* in the same line produced so that all the vectors will represent pulls in the diagram. This convention should be followed in order to avoid confusion in drawing the force diagrams. Obviously the *effect* on the ring, in equilibrium at B, is the same whether there is a horizontal push from the left or a horizontal pull of the same amount to the right.

The graphical solution can be as follows: Draw a direction diagram of the apparatus, using a *direction* scale of say $\frac{1}{8}$ in. = 1 in.; then the vertical line AC will be 3 in. long and AB will also be 3 in. Produce AB to the right an indefinite length for the force line of AB; also draw a vertical line for the weight W. From B, using a *force* scale of say $\frac{1}{16}$ in. = 1 lb, draw BR equal and opposite to the weight W; this will be vertically up $2\frac{3}{16}$ in. Resolve this into its components BS and BT by completing the parallelogram on BR as a diagonal.

Scale $BT = 3\frac{3}{32}$ in. = $49\frac{1}{2}$ lb, the tension in the tie BC, *Ans.* (a).

Also scale $BS = 2\frac{3}{16}$ in. = 35 lb, the compression in the stick AB and consequently its thrust against the wall at A, *Ans.* (b).

PROBLEMS

3–1. Determine whether the following forces on a point are in equilibrium: *A* due east 30 lb, *B* due north 40 lb, *C* 50 lb acting southwest at an angle of 37° with the south.

3–2. Forces of 15 and 27 lb act at an angle of 60°. What third force acting with these will produce equilibrium?

3–3. Forces of 100 and 125 lb act at an angle of 50°. Find their equilibrant.

3–4. A tight-rope walker weighing 150 lb stands in the middle of a tight rope 24 ft long. The rope is depressed 1 ft. Find the tension in the rope.

3–5. In a simple roof truss the span *AC* is 20 ft, each rafter *BA* and *BC* has a pitch of 1 ft vertical in each 3 ft horizontal, and a load of 1000 lb acts at the peak *B*. (*a*) Find the force in each rafter due to the load. (*b*) Find the horizontal and vertical components of the thrust in rafter *AC* on the point *C*.

3–6. In a simple stick-and-tie arrangement, similar to Fig. 40, the stick *AB* is 24 in. long and is held at an angle by the tie *BC* which is 21 in. long. The tie is fastened at *C* which is 30 in. above *A*. If the load is 10 lb at *B*, find the tension *T* in the tie, and the compression *C* in the stick.

3–7. A rod *AB*, hinged to a wall at the end *A*, is held inclined 40° to the vertical by a cord running horizontally from the end *B* to a point on the wall vertically above the hinge. A weight of 16 lb is hung from the end *B*. Find the tension in the cord and the thrust of the stick against the wall at the hinge.

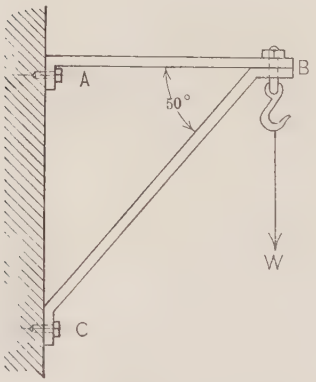


Fig. 38.

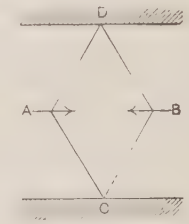


Fig. 39

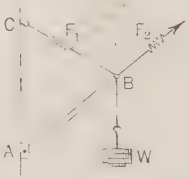


Fig. 40.

3–8. If the breaking strength of the cord in Problem 3–7 is 35 lb, what weight hung at *B* will be just enough to break the cord?

3–9. A bracket similar to Fig. 38, but with a slightly different angle, supports a load of 325 lb on the hook. If *AB* measures 20 in. and *AC* 25 in., solve for the compression *C* in *BC* and the tension *T* in *AB*.

3–10. The anchor rope of a balloon makes an angle of 70° with the ground. If the "lifting force" of the balloon is 250 lb, find the tension in the anchor rope and the horizontal force exerted against the balloon by the wind.

3–11. A barrel weighing 296 lb rests on a plank inclined 27° to the horizontal. Neglecting friction, what force parallel to the plank will be required to prevent the barrel from rolling down the plank? What is the force of the barrel at right angles to the plank?

3–12. $ABCD$, Fig. 39, is a frame of four sides having equal lengths and pivots at the four corners, such as the arrangement on some automobile jacks. If the angle ACB is 60°, find the thrust in each arm when A and B are drawn together by a force of 15 lb and find also the force separating C and D.

24. Three concurrent forces in equilibrium, acting at different points on a body. Since the point of application of a force may be regarded as at any point in the line of action of the force, the solution given in the preceding article is also applicable to bodies acted upon by three forces in the same plane, not parallel, and applied at different points.

Thus in the illustration of Fig. 41, in which AB is any heavy body (e.g., a bar, picture, or other object), supported by flexible cords, AC and BC, which are not parallel, three forces act upon AB producing equilibrium. These are: The weight W of AB acting at G, its center of gravity (Art. 9), and the two pulls applied by AC and BC respectively.

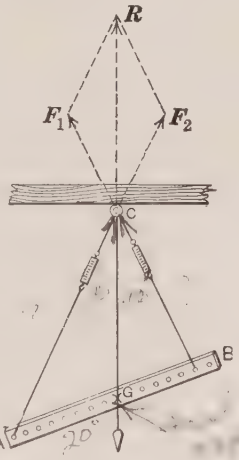


FIG. 41. A suspended body in equilibrium.

The forces in AC and BC lie along the cords, and therefore their lines of action intersect at the point C. They are equivalent to a single force (their resultant) acting through C, and hence, if the third force acting on AB (its weight W) is to produce equilibrium with the forces AC and BC, its line of action must also pass through C and it must be equal in amount to the resultant of the first two. The solution is as suggested in Fig. 41. No matter what the lengths of the supporting cords may be, the *center of gravity of the body must lie in a vertical line through their intersection.*

In general, **no matter where on the body the forces are applied, if it is in equilibrium under the action of three nonparallel forces in one plane, the lines of action of the forces must pass through a common point and constitute a system of concurrent forces.**

If the cords at A and B are divergent, their force lines *produced* will intersect *below* the bar. However the construction of the force parallelogram at the new point C will be similar to that in Fig. 41.

A common example of equilibrium with three forces is furnished by the bent-arm lever used in operating switches, semaphores, etc., on railroads. A pull F_1, Fig. 42, is applied to rotate the lever; this gives rise to a force F_2, which operates the signal. At the same time a force is set up against the pin at B about which the lever rotates. At any given position, the lever is held in equilibrium (neglecting friction) by three forces, F_1, F_2, and the reaction R of the pin. Any one being known, the others may be determined.

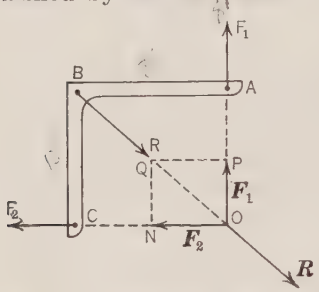


Fig. 42. Forces on a bent-arm lever.

The construction of the force parallelogram will be clear from the figure, in which the three forces are represented by the vectors $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{R}$. (In this case $\mathbf{R}$ is equal and opposite to the resultant OQ in the parallelogram.)

PROBLEMS

3–13. If AB, Fig. 41, is a uniform metal bar weighing 20 lb, and a plumb line hanging from C, the point of support, makes an angle of 30° with the cord CA, and an angle of 40° with the cord CB, find the tensions in CA and CB.

3–14. A bar AB weighing 20 lb is supported by two cords as in Fig. 49. The cords make angles of 51° and 48°, respectively, with the horizontal. Find the tension in each cord.

3–15. If in the bent lever, Fig. 42, BC is 10 in. long, and is at right angles to BA, which is 8 in. long, and if force F_1 of 8 lb is parallel to BC and force F_2 is parallel to BA, find the force F_2 and the direction and magnitude of the reaction at the pin B.

3–16. A timber, 16 ft long and weighing 128 lb, leans against a vertical wall as AB in Fig. 43. The weight acts at the center of the timber, which leans at at angle of 70° with the ground at A. Assuming that there is no friction at the end against the wall, find the horizontal reaction at the wall, and the amount and direction of the ground reaction at A.

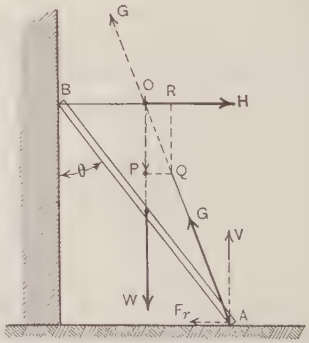


Fig. 43. Forces on a timber leaning against a smooth wall.

3–17. What horizontal force at the axle will be required to draw a wheel of radius $1\frac{1}{2}$ ft and weight 200 lb over an obstacle 6 in. high? (When free from the ground, three forces act on the wheel: its weight, the horizontal pull, and the reaction of the obstacle.)

25. The force triangle. The force triangle described in Art. 19 is convenient as a test for the equilibrium of three concurrent forces. Reference to Fig. 30 shows that the three forces, **b**, **a**, and **c**, are equal in magnitude and parallel in direction to the three sides of the triangle ONR. The direction of the arrow heads is such that they follow one another the *same way around* the triangle, Fig. 44.

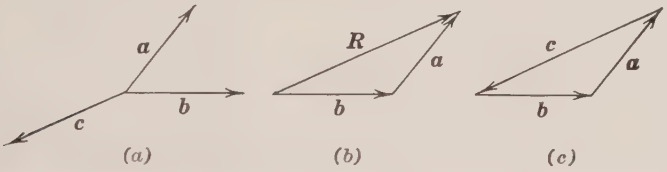


FIG. 44. Equilibrium by the force triangle. **c** is equal and opposite to **R**.

The resultant of **a** and **b** is a force acting from the origin of **b** to the forward end of **a** (Art. 19). The *equilibrant* of **a** and **b** is equal and opposite to the resultant and equals the force **c** acting from the forward end of **a** to the origin of **b**.

Therefore: **if three forces in equilibrium be drawn to scale in amount and in proper direction, they will form a triangle, with the arrow heads pointing the same way around.**

Example 1.

Solve the simple truss of Art. 23, Fig. 37.

Solution (*a*). Draw the direction diagram as in Art. 23. From the point B on the direction diagram draw the vector **W**, acting down; from its forward end draw a horizontal line parallel to AB intersecting the production of CB at M. See Fig. 45. Place arrows on the vectors in the same direction around the triangle. Scale **C** to determine the force in the stick, and **T** for the force in the tie.

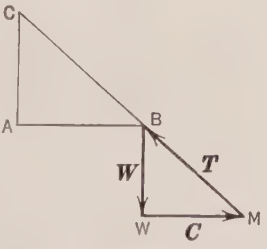


FIG. 45. Solution of Example 1 by the force triangle.

Solution (*b*). The forces bear the same relation to each other as the sides of the apparatus triangle ABC to which they are parallel. That is W: ten-

sion in BC: thrust in BA = side CA : side CB : side BA. $W = 35$ lb; sides CA and BA are each 24 in. from which

side $CB = \sqrt{24^2 + 24^2} = \sqrt{(576 + 576)} = \sqrt{1152} = 34$ in., about

then
$$\frac{35}{24} = \frac{\text{tension in } BC}{34}$$

$$\text{tension in } BC = \frac{35 \times 34}{24} = 49.5 \text{ lb} \quad Ans.$$

and
$$\frac{35}{24} = \frac{\text{thrust in } BA}{24}$$

$$\text{thrust in } BA = \frac{35 \times 24}{24} = 35 \text{ lb} \quad Ans.$$

Example 2.

It is required to find the angles at which forces of 10, 15, and 20 lb must act to produce equilibrium.

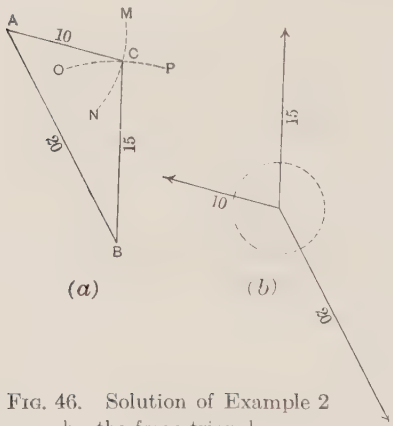


FIG. 46. Solution of Example 2 by the force triangle.

Solution. First lay off a line AB to represent a force of 20 lb. From A with a radius to represent 10 lb, describe an arc MN, and from B with a radius to represent 15 lb describe an arc OP to intersect this at C. Then AB, BC, and CA, Fig. 46(a), represent the directions of the forces, and Fig. 46(b) shows the angles between them.

26. The force polygon. When a system of concurrent forces includes more than three forces the force polygon described in Art. 19 offers a simple graphical solution.

If any number of concurrent forces in equilibrium in the same plane be drawn as vectors, each starting from the forward end of the preceding one, they will form a closed polygon, with the arrow heads pointing the same way around. See Fig. 47.

If the polygon of a given force system does *not* close, the forces are not in equilibrium; the *equilibrant* will then be the vector necessary to close, with the arrow heads pointing the same way around. *One unknown force* is determined in this manner.

Two forces of unknown amount, but of known direction, also may

be determined by the force polygon. Draw the known vectors in any order, and complete the polygon with two lines parallel to the

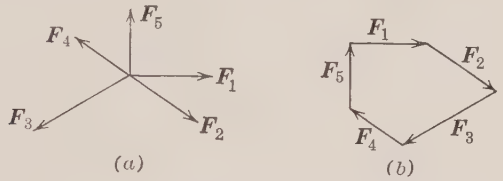


FIG. 47. (*a*) Five forces in equilibrium.
(*b*) The force polygon closes.

unknowns, one line from each end of the known series. They should intersect each other. Place arrow heads to point the

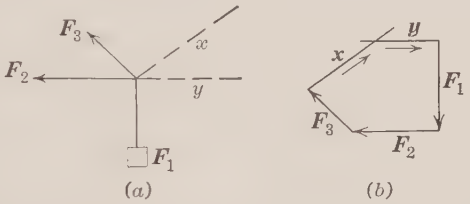


FIG. 48. Solution for two unknown forces in (*a*) by the force polygon in (*b*).

same way around, thus determining the desired forces in amount and direction. Thus in Fig. 48 forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ are known, and **x** and **y** are to be determined.

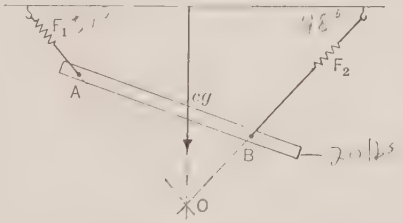


FIG. 49. A suspended bar in equilibrium.

PROBLEMS

3–18. Will forces of 4, 5, and 10 lb acting at a point produce equilibrium?

3–19. Find the angles at which the concurrent forces 35, 50, and 60 lb must act to produce equilibrium.

3–20. Given forces of 4, 5, and 7 lb in equilibrium at a point; find the angle between the 4- and 5-lb forces.

3–21. Given the following forces at a point, find the equilibrant; 50 lb horizontal toward the right, 60 lb 30° above the horizontal toward the right, and 70 lb vertical.

3–22. Find the equilibrant of the following four forces at a point: 55 lb horizontal, 45 lb above the horizontal 35°, 30 lb below the horizontal 40°, 20 lb below the horizontal 60°; all acting to the right from the point.

SUMMARY

Equilibrium is a condition in which the forces acting on a body are balanced. When a body is at rest, or moving uniformly without change, it is said to be in equilibrium.

The two conditions of equilibrium are (1) no tendency toward motion of translation and (2) no tendency toward motion of rotation.

Concurrent forces are those whose lines of action meet at a common point, whether or not they act at the same point on a body.

Two forces will cause equilibrium if they are equal in amount and opposite in direction; in which case their resultant equals zero.

Three forces will cause equilibrium if the diagonal of the force parallelogram formed on any two forces as sides is equal and opposite to the third force; also if, when drawn to scale in amount and in proper direction, they form a **force triangle** with the arrow heads pointing the same way around.

Any number of forces will cause equilibrium if, when drawn as vectors, each starting from the forward end of the preceding one, they form a closed **force polygon** with the arrow heads pointing the same way around.

An equilibrant is a force which, acting jointly with the other forces in the system, will produce equilibrium. It is equal and opposite to the resultant of the other forces; also it is the force which will close the triangle or polygon constructed with the other force vectors.

One unknown force in a system of concurrent forces in equilibrium can be determined by finding the equilibrant of the known forces.

Two unknown forces in a system of three concurrent forces in equilibrium can be determined by resolving the equilibrant of the known force (equal and opposite to it) into components along the required directions, by use of the parallelogram. If there are three or more than three, it can be accomplished, as described in Art. 26, by use of the polygon.

The direction of the forces is given, sometimes by their angles with each other or with the horizontal or vertical and sometimes by the dimensions of the structure in which they act. In the latter case the construction of a **direction diagram** to scale is necessary, upon which the force diagram can be superimposed.

SYMBOLS AND ABBREVIATIONS

Same as in preceding chapters. See complete list in Appendix.

CHAPTER IV

MATHEMATICAL SOLUTION OF CONCURRENT FORCES

This chapter treats of the mathematical solution of systems of concurrent forces in one plane, including the computation of the resultant of the components, or of the equilibrant as required.

In Chapter II the resultant of two or more forces was determined graphically by the use of vector diagrams, and also a given force was resolved into components by similar methods. In Chapter III these principles were used to analyze a system of concurrent forces in equilibrium; that is to determine an unknown force in such a system, or the equilibrant force to be applied to cause equilibrium. The accuracy with which a resultant or an equilibrant can be determined graphically is dependent upon the accuracy with which the force diagram can be drawn and measured. A more precise and convenient method is furnished by trigonometry.

For the benefit of the student who has not studied trigonometry, a brief explanation of the meaning and use of the few simple trigonometric functions required for this purpose, and of one or two useful formulas, is given in the Appendix. Experience has shown that a working knowledge of this method can soon be acquired, and that time will be saved in the end by stopping here long enough to study the topics and follow the brief drill suggested in the Appendix; see pages 342 to 347.

In the following articles two equations for equilibrium will be developed, which will not only satisfy the conditions for this first group of concurrent forces, but can also be used in connection with parallel forces in Chapter V and non-concurrent forces in Chapter VI.

27. Composition of forces in the same straight line. The mathematical solution is similar to that for the composition of velocities or any vector quantities discussed in Art. 12.

If the component forces all act in the same straight line, the resultant force equals their *algebraic* sum. Its direction is determined by the algebraic sign. Forces acting vertically upward are

usually regarded as *positive*; forces acting downward, as *negative*. Similarly horizontal forces toward the right are usually regarded as *positive*, and those toward the left as *negative*. The **equilibrant** is equal to the resultant but acts in the opposite direction.

Example.

Find the resultant of the following forces all acting at a point O: 10 lb up; 6 lb up; 7 lb down; 2 lb down. Also find the equilibrant of this system.

Solution.

$$+10 + 6 - 7 - 2 = +16 - 9 = +7$$

The resultant = 7 lb, acting up. *Ans.*
The equilibrant = 7 lb, acting down. *Ans.*

28. Composition of forces at an angle. Computation of resultant. Reference to Fig. 18 shows that, since BR is equal and parallel to OA (which represents the force $\mathbf{F}_1$), the two forces and

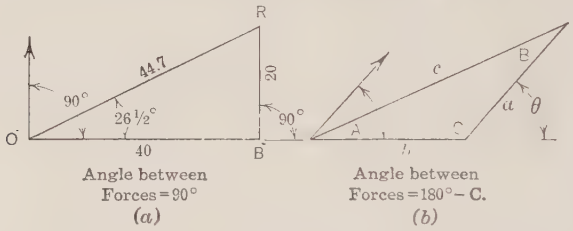


Fig. 50.

their resultant form a triangle OBR (one-half the parallelogram). If $\mathbf{F}_1$ and $\mathbf{F}_2$ act at right angles, the solution of a simple right triangle only is required. See Appendix, Art. 6.

Example 1.

Given a horizontal force of 40 lb and a vertical force of 20 lb; compute their resultant, and its angle with the horizontal.

Solution. In the *right* triangle OBR, Fig. 50(a), side $OB = 40$, $BR = 20$, and (by geometry)

$$OR = \sqrt{(BR^2 + OB^2)} = \sqrt{(20^2 + 40^2)} = 44.7+$$

Also (by trigonometry)

$$\tan \text{ angle } ROB = \frac{BR}{OB} = \frac{20}{40} = 0.500$$

Then angle $ROB = 26\frac{1}{2}°$ approximately (see tables)

The amount of the resultant can *also* be found by the trigonometric equation:

$$\sin ROB = \frac{BR}{OR}$$

Therefore $$OR = \frac{BR}{\sin 26\frac{1}{2}^\circ} = \frac{20}{0.446} = 44.7+$$

That is, the resultant of the 40- and 20-lb forces is a force of about 44.7 lb acting at an angle of about $26\frac{1}{2}^\circ$ above the horizontal. *Ans.*

These answers may be compared with the graphical solution of Art. 14. Additional illustrative examples are given in the Appendix in Art. 6.

If the angle between the component forces is other than 90°, the oblique triangle can be solved by the two general equations known as the *law of cosines* (used to determine the amount of the force), and the *law of sines* (used to determine the angle of direction). The solution of a problem by these equations is given in the Appendix, Art. 6.

This type of problem can also be solved by a *formula derived from these two laws.* The law of cosines is, using the symbols of Fig. 50(*b*), $c^2 = a^2 + b^2 - 2\,ab \cos C$. The law of sines is $\sin A / \sin C = a / c$. Observe that c is the resultant of a and b; also that angle θ, between the components, equals the external angle at C, and the interior angle C equals $180^\circ - \theta$. Therefore $\cos C$ equals $-\cos \theta$, because the cosines of obtuse angles are negative. Then, substituting R (resultant) and θ for c and C respectively, the law of cosines can be stated, thus:

$$R^2 = a^2 + b^2 + 2\,ab \cos \theta$$

Notice particularly the difference in sign of the cosine term, a frequent cause of confusion between the trigonometric law and the mechanics formula. If the interior angle at C is used, the sign is $-$; if the external angle θ is used, it is $+$. Again, substituting $180^\circ - \theta$ for angle C, the law of sines becomes $\sin A = \sin (180 - \theta) \times \frac{a}{c}$

Example 2.

Two forces of 50 lb and 60 lb, respectively, act at an angle of 40°; compute their resultant.

Solution.

$$R^2 = 50^2 + 60^2 + 2 \times 50 \times 60 \times 0.766$$

$$R = \sqrt{10{,}696} = 103 \text{ lb} \quad \textit{Ans.}$$

and $$\sin A = 0.643 \times \frac{50}{103} = 0.312$$

$$A = 18.2° \quad Ans.$$

Therefore the resultant is a force of 103 lb acting about 18° from the 60-lb force. *Ans.*

For a more detailed solution see the Appendix, Art. 6.

29. Computation of rectangular components. The resolution of forces shown graphically in Art. 17 can also be accomplished mathematically.

Thus, referring to Fig. 25: $\frac{OX}{OR} = \cos \theta$

Therefore $OX = OR \cos \theta$

Similarly $\frac{XR}{OR} = \sin \theta$

Therefore $XR = OY = OR \sin \theta$

Note that when the angle between the force and the horizontal is given, **the H component equals force times cos angle**; and **the V component equals force times sin angle.**

Example.

A force of 80 lb acts at an angle of 30° to the horizontal. Find the values of its horizontal and vertical components.

Solution. From the table of trigonometric functions, sin 30° equals 0.500 and cos 30° equals 0.866. Therefore,

horizontal component $= 80 \times 0.866 = 69.3$ lb. *Ans.*
vertical component $= 80 \times 0.500 = 40.0$ lb *Ans.*

See Example 6 in Appendix, Art. 6.

30. Composition of several forces by the method of resolution along rectangular axes. When the resultant of several forces is desired, there is a simpler method than the solution of separate triangles for each pair of forces.

The simpler method consists in resolving each of the given forces into its H and V components, giving a system of forces all of which are either horizontal or vertical. This system is equivalent to, and can replace the original system. Then find the sum of all H components; and the sum of all V components. These

two sums are also equivalent to, and can replace, the original system. Then the resultant of these two forces will be the resultant of the original system.

This method, known as the method of H and V components, can be performed either graphically or mathematically. The use of squared paper is a convenience in the graphical procedure.

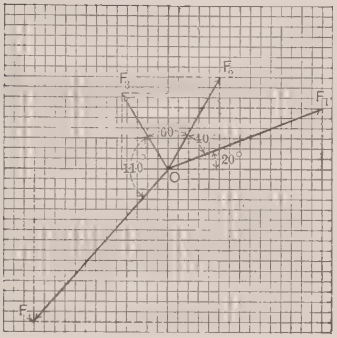


Fig. 51.

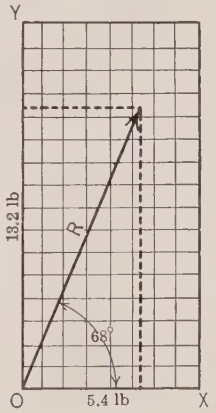


Fig. 52.

Example.

Suppose that four forces act at a point O. Force F_1 of 32 lb acts at an angle of 20° with the horizontal; force F_2 of 20 lb acts at an angle of 40° with F_1; force F_3 of 18 lb acts at an angle of 60° with F_2; and force F_4 of 40 lb acts at an angle of 110° with F_3. Required to find the amount and direction of their resultant.

Solution. Determine the H and V components of each force. The H components (using the angles with the horizontal in each case):

of $F_1 = 32 \cos 20° = 32 \times 0.940 = 30.1$ lb (to the right)
of $F_2 = 20 \cos 60° = 20 \times 0.500 = 10.0$ lb (to the right)
of $F_3 = 18 \cos 60° = 18 \times 0.500 = 9.0$ lb (to the left)
of $F_4 = 40 \cos 50° = 40 \times 0.643 = 25.7$ lb (to the left)

The V components:

of $F_1 = 32 \sin 20° = 32 \times 0.342 = 10.9$ lb (up)
of $F_2 = 20 \sin 60° = 20 \times 0.866 = 17.3$ lb (up)
of $F_3 = 18 \sin 60° = 18 \times 0.866 = 15.6$ lb (up)
of $F_4 = 40 \sin 50° = 40 \times 0.766 = 30.6$ lb (down)

The resultant of the H components

$= +30.1 + 10.0 - 9.0 - 25.7 = +5.4$ lb (to the right from O)

The resultant of the V components

$$= +10.9 + 17.3 + 15.6 - 30.6 = +13.2 \text{ lb (up from } O)$$

Now determine the resultant of these resultants:

$$\text{tan angle [with horizontal]} = \frac{13.2}{5.4} = 2.444$$

Then angle with horizontal = 68° (nearly, see tables)

And $$\text{resultant} = \frac{13.2}{\sin 68°} = \frac{13.2}{0.927} = 14.3 \text{ lb}$$

That is, the resultant of the four forces is about 14.3 lb acting at an angle of about 68° above the horizontal. *Ans.*

The graphical procedure is suggested in Fig. 51 and Fig. 52.

PROBLEMS

4–1. Find the resultant mathematically of 50 lb acting horizontally and 31 lb acting vertically.

4–2. Find the resultant of 70 lb south and 130 lb west.

4–3. Find the resultant of 180 and 273 lb at an angle of 90°.

4–4. Find the resultant of 24 and 36 lb acting at an angle of 30° with each other.

4–5. Find the resultant of 50 and 75 lb acting at an angle of 70° with each other.

4–6. Find the resultant of 200 and 300 lb acting at an angle of 120° with each other.

4–7. Resolve a force of 700 lb acting upward and to the right at an angle of 25° with horizontal into its horizontal and vertical components.

4–8. A freight car is being drawn along the track by a horse walking beside the track. The chain by which the horse is attached is horizontal, but makes an angle of 30° with the rails. If the tension in the chain is 350 lb, what is the force drawing the car along the rails? The component of the tension producing pressure of the wheel flanges against the rail?

4–9. A canal boat is towed by a pull of 130 lb at an angle of 15° with the axis of the boat. Find

(*a*) The force urging the boat directly ahead in the line of its axis and

(*b*) The force urging the boat toward shore.

4–10. Find by the method of H and V components the resultant of 200 lb vertically down, 140 lb horizontally right, and 160 lb downward to the right at 60° to the vertical. Compare answer with the graphical solution of Problem 2–20 in Chapter II.

4–11. Find by the method of H and V components the amount and direction of the resultant of the combination of forces shown in Fig. 53(*a*).

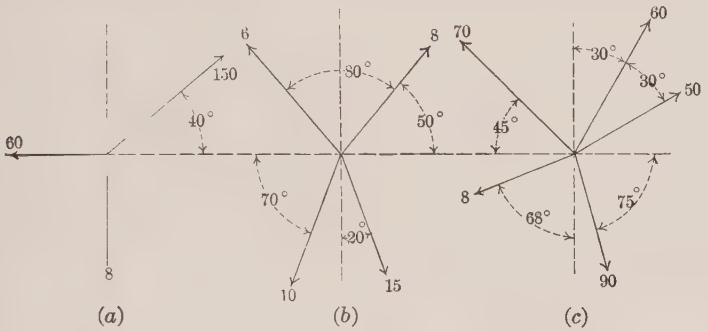


FIG. 53.

4–12. Find the resultant for Fig. 53(*b*).

4–13. Find the resultant for Fig. 53(*c*).

4–14. If the total resultant force on an airplane wing, considered acting at its center of pressure, is 1500 lb at an angle of 4° with the vertical, find the total vertical lift L, and the total horizontal drag D; also the lift-drag ratio L/D.

31. Mathematical conditions of equilibrium. As stated in Art. 21, in order that a given force system be in equilibrium it is necessary that there shall be no tendency to produce (1) motion of translation, or (2) motion of rotation. It has been shown that all forces in a given system can be resolved into rectangular com-

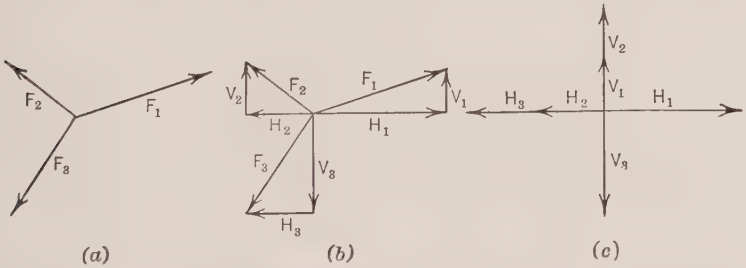


FIG. 54. Three forces in equilibrium (*a*), resolved into H and V components (*b*), $H_1 = H_2 + H_3$ and $V_1 + V_2 = V_3$ (*c*).

ponents along horizontal and vertical axes, giving two systems which together can replace the original. One system will consist entirely of horizontal, and the other entirely of vertical forces.

If the resultant of each of these systems equals zero, then *the resultant of the whole system of concurrent forces must equal zero and there will be no tendency toward motion of translation.*

A body acted upon by any number of concurrent forces in the same plane will be in equilibrium when, if the forces be resolved into their H and V components, the resultants of both the H and V components equal zero. Fig. 54.

Let ΣH represent the algebraic sum of all horizontal components of the original forces, and ΣV the algebraic sum of all the vertical components. Then the mathematical conditions of equilibrium for concurrent forces are:

$$\Sigma H = 0 \tag{1}$$

$$\Sigma V = 0 \tag{2}$$

The convention used in preceding articles that forces to the right are to be considered positive, forces to the left negative, forces upward positive, and forces downward negative, is followed here and throughout the text.

Example 1.

For the simple truss of Fig. 37, the forces in equilibrium at B are as shown in Fig. 55, not necessarily drawn to scale, where T represents the tension in the tie, C the thrust of the rod at B, and W the load.

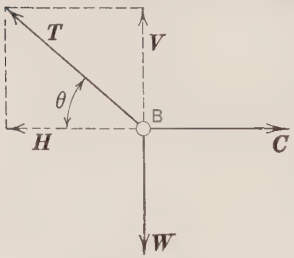


Fig. 55.

Solution. The tension T in the tie BC may be resolved into its horizontal and vertical components H and V, where $H = T \cos\theta$ and $V = T \sin\theta$. Then, the equations for equilibrium are:

$$\Sigma H: \quad C - T\cos\theta = 0 \tag{1}$$

$$\Sigma V: \quad T\sin\theta - W = 0 \tag{2}$$

Assume the case in which W is 50 lb and the angle θ 40°. Substituting these values in equation (2) gives

$$T \sin 40° - 50 = 0$$

Hence, $T = \dfrac{50}{\sin 40°} = \dfrac{50}{0.643} = 77.8$ lb; or the tension in the tie BC is 77.8 lb. *Ans.*

This value, substituted in equation (1), gives

$$C - 77.8 \cos\theta = 0$$

Or $$C = 77.8 \times 0.766 = 59.6 \text{ lb} \quad \textit{Ans.}$$

This force, 59.6 lb, is the force with which the horizontal rod pushes to the *right* at B. It is also the push to the *left* at A, and is therefore the compression in the member AB. See Fig. 37.

Example 2.

In the device shown in Fig. 56(a), suppose the necessary angles and the load at B are given; find the stresses in CB and AB.

Solution. Construct the force diagram for point B, not to scale, as shown in Fig. 56(b). Let T represent the pull exerted by the tie AB, and C the thrust of the stick CB.

(If the student has any difficulty in understanding why T and C are given the directions they here have, let him consider: (1) the effect if AB be cut; (2) the effect if AB be replaced and then CB be cut. In what direction is each force acting upon the point B?)

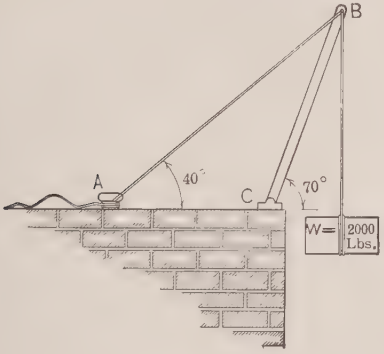


Fig. 56(a).

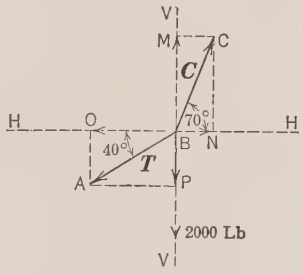


Fig. 56(b).

Resolve C and T into their horizontal and vertical components and write the two equations referring to Fig. 56.

$$\Sigma H: \qquad BN - BO = 0 \tag{1}$$

$$\Sigma V: \quad BM - BP - 2000 = 0 \tag{2}$$

But

$$BN = C \cos 70° = 0.342\ C$$

$$BM = C \sin 70° = 0.940\ C$$

$$BO = T \cos 40° = 0.766\ T$$

$$BP = T \sin 40° = 0.643\ T$$

Substituting these values in equations (1) and (2), we have the equations:

$$0.342\ C - 0.766\ T = 0 \tag{1a}$$

$$0.940\ C - 0.643\ T - 2000 = 0 \tag{2a}$$

Solving the simultaneous equations (1*a*) and (2*a*)* we obtain the values,

$$T = 1370 \text{ lb}, \quad C = 3070 \text{ lb} \quad Ans.$$

Example 3.

Required to find the tension in BD and the compression in BA of the wharf crane shown in Fig. 57 due to a load at W of one ton, when the members are in the positions indicated. A force diagram for the point

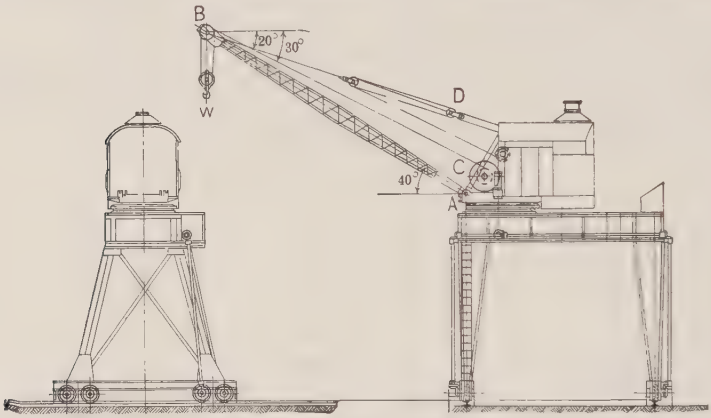


Fig. 57. Wharf crane.

B is as shown in Fig. 58, the letters given to the forces of the diagram indicating the members in which they occur and the arrow heading the directions of the forces with respect to the point B. Note that force BC is known, $= \frac{1}{2}$ W, as one of the two ropes holding the load passes over the pulley at B. See Art. 43.

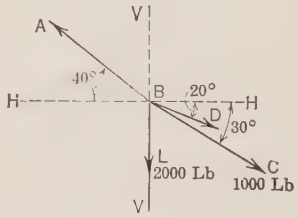


Fig. 58.

Solution. The two equations for equilibrium are:

$$\Sigma H: \quad BD \cos 20^\circ + 1000 \cos 30^\circ - BA \cos 40^\circ = 0 \qquad (1)$$

$$\Sigma V: \quad BA \sin 40^\circ - 2000 - 1000 \sin 30^\circ - BD \sin 20^\circ = 0 \qquad (2)$$

* See solution of these simultaneous equations in Appendix, Art. 7.

Substituting the values of the functions of the angles, transposing and collecting terms, gives

$$0.940\ BD - 0.766\ BA = -866 \qquad (1a)$$

$$-0.342\ BD + 0.643\ BA = 2500 \qquad (2a)$$

From which* we find the tension in BD to be 3980 lb and the compression in BA to be 6010 lb. *Ans.*

PROBLEMS

4–15. In a simple "stick and tie" arrangement the stick is inclined 60° to the vertical, the tie is 45° to the vertical, and the load at the end of the stick is 120 lb. Find the forces in stick and in tie.

4–16. In a simple roof truss a weight of 800 lb is supported at the peak of two rafters AB and AC, each making 60° with the vertical.

(*a*) Find the force in each rafter due to the 800 lb load.

(*b*) Find the horizontal and vertical components of the thrust of rafter AC on the point C.

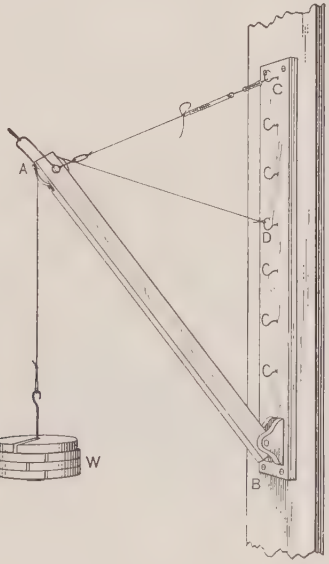


FIG. 59. Laboratory model of a hoisting crane.

4–17. The bracket of Fig. 38 is supporting a load W of 72 lb. What is the tension in AB and the compression in CB?

* See Appendix, Art. 7.

4–18. A wall crane as in Fig. 96 has a load at B of 1000 lb. The angle at B is 32°. Find the tension in the wire BC due to the load; also the compression in the beam AB.

4–19. A pull F of 20 lb at an angle of 15° to the plane AB is just sufficient to hold the roller in an apparatus similar to Fig. 36. Plane AB is inclined at an angle of 50°. Compute the weight of the roller and the normal component, that is, the force perpendicular to the plane.

4–20. A body weighing 500 lb, suspended from a rope, is pulled aside until the suspension rope is 10° out of vertical. What will now be the tension in the supporting rope? What *horizontal* pull is required to hold the body with the rope at this angle?

4–21. What will be the forces in AB and BC, Fig. 56(*a*), if, in addition to the 2000-lb load, there is a horizontal pull to the right at B of 750 lb?

4–22. Find the forces in the members of the hoisting crane model, Fig. 59, when the load $W = 30$ lb, $\angle ABC = 40°$, $\angle CAW = 120°$, and $\angle CAD = 50°$.

4–23. (*a*) A derrick, or hoisting crane, is rigged about as the model in Fig. 59. The mast BC is 25 ft high, the jib or boom BA is 30 ft, and makes an angle of 50° with the horizontal, the point D is 2 ft above B. Compute the tension T in AC and the compression C in AB due to a 1-ton load.

(*b*). Solve 4-23 (*a*) by the graphical method of the force polygon as a check on the computed answers.

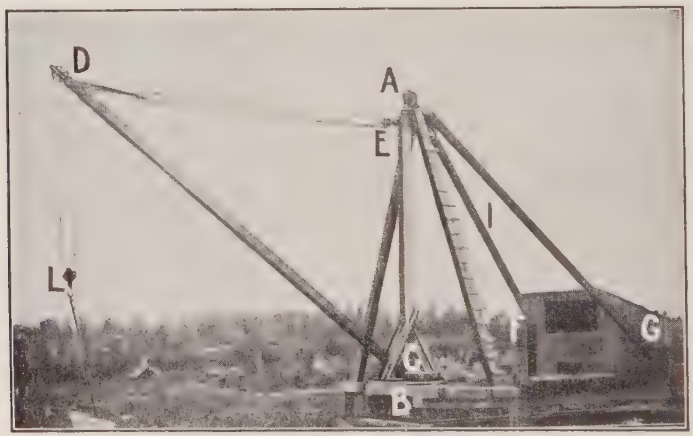


FIG. 60. Derrick car.

4–24. Compute the tension in DE and the compression in DC of the derrick in Fig. 60, due to a load L of 3000 lb. Assume that the hoisting rope from L passes over the pulley and runs parallel to DC. Assume also that DE lies 20° below the horizontal and that the angle LDC equals 50°.

SUMMARY

The resultant of forces in the same straight line equals their algebraic sum. Forces upward are positive, downward negative; forces toward the right are positive, toward the left negative.

The equilibrant is equal and opposite to the resultant.

The mathematical composition and resolution of forces is accomplished by the solution of the triangle which comprises one half the force polygon. Forces at right angles require only the solution of a right triangle. Forces at other than 90° require the solution of an oblique triangle or can be solved by the method of H and V components, employing only right triangles. The procedure is to sketch the force diagram or triangle, and select the trigonometric equation and functions which include the known forces and the desired unknown, substitute the known values, and solve. See Art. 28 and Art. 29. See Trigonometry, in the Appendix.

The resultant of any number of forces can be found by the resolution of the forces into their **horizontal and vertical components,** then finding the sum of the H components and of the V components, and then the resultant of these two sums.

The mathematical conditions for equilibrium of concurrent forces are stated in the two equations:

$$\Sigma H = 0 \tag{1}$$

$$\Sigma V = 0 \tag{2}$$

These equations are sufficient for motion of translation and for concurrent forces.

The procedure in determining one or two unknown forces in a system in equilibrium is to draw or sketch the force diagram, with all the forces acting as pulls from the point to avoid confusion. Then resolve all forces into their H and V components, and set up two equations, one for $\Sigma H = 0$ and one for $\Sigma V = 0$. Solve these equations by algebra. See suggestions in the Appendix.

CHAPTER V

MOMENTS. PARALLEL FORCES

The subjects to be studied in Chapter V are, first, the moment of a force and torque. These terms were defined in Art. 10. An explanation of their meaning and of the law of moments is given. The law of moments is a means of determining whether a body is in equilibrium as regards motion of rotation. The principle of moments has many applications, some of which are treated.

The second group of forces, that of parallel forces in one plane, is next considered. These can be solved by the application of the law of moments, or moment equation, together with one of the two equations for equilibrium used in the preceding chapter. The method of determining the resultant and the equilibrant of systems of parallel forces is included.

In this chapter also is the use of the law of moments to determine the location of the center of gravity of bodies and surfaces. This computation is a preliminary step in the solution of problems in strength of materials in a later chapter.

32. Moment of force. The tendency of a force to produce rotation about a given axis is called the **moment of the force** with respect to that axis.

One of the effects which may be produced by the application of force to a body has been stated to be change in the existing motion of the body. This change may be change in motion of translation or change in the rotation of the body, depending upon the conditions and upon the direction of the force and the point at which it is applied. Thus a pull on the sled in Chapter I may cause motion of translation; a pull on a rope wound around the rim of a wheel may cause rotation.

A force applied to a shaft or wheel may or may not tend to produce rotation. Thus consider the drive wheel in Fig. 61. When the connecting rod is in the position PR, the force along it tends merely to push the wheel against the axle X, since the force line MN of the push exerted by the connecting rod passes through the center of the axle. Any amount of force exerted by the rod, under this condition, will not cause the wheel to rotate.

Let the pin, however, move to the new position P', where the force line of the thrust no longer passes through the axis, and it is apparent that a push exerted by the rod will tend to produce rotation. The force now has a lever arm upon which it may act. The length of this lever arm is the length of the perpendicular a

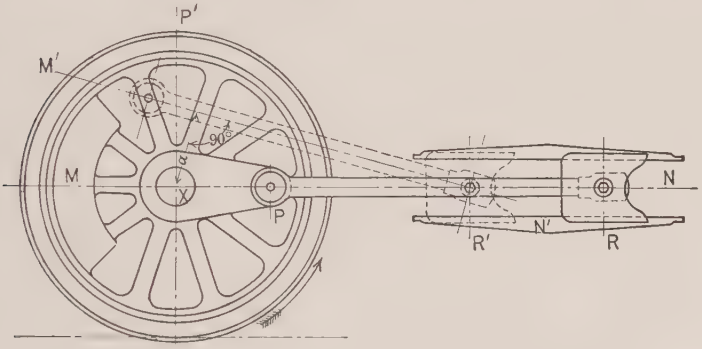


Fig. 61. Moment of force exerted by a connecting rod.

to the action line of the force, extending from the action line to the axis of rotation.

This perpendicular from the action line of a force to the axis of rotation is called the **moment arm** of the force. A force having a moment arm with respect to an axis is said to have a **moment** about that axis.

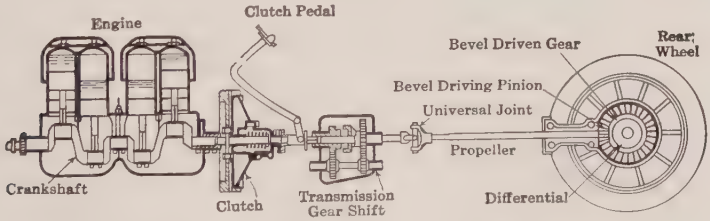


Fig. 62. Moments of force on an automobile crankshaft.

The crankshaft of an automobile engine, Fig. 62, while one solid forging, is not straight, but is offset where the connecting rods leading to the cylinder heads are attached. These offsets provide moment arms for the forces exerted along the connecting rods. The moment arms vary in length with the positions of the connecting rods as the crankshaft rotates, but are, at any instant, the lengths of the perpendiculars to the axes of the connecting

rods measured from these axes to the axis about which the crankshaft turns.

Moment of force is also called **torque** when applied to shafts and pulleys.

33. Measure of moment or torque. The amount of the moment of a force depends upon both the *amount of the force* and the *length of the moment arm.* This may be seen from a consideration of the apparatus of Fig. 63 in which PQ is a stiff rod balanced about a pin O, and free to rotate in either direction about the pin. If now we hang a weight W, of 10 lb at M, 5 in. from the pin, this force will turn the rod in a *clockwise* direction about the pin. To prevent this we must have an *equal moment counter-clockwise,* and if this is furnished by a pull F applied by a spring balance, also attached 5 in. from O, we shall find this balance will read 10 lb (the same as W). (2) If W is now moved to P, 10 in. *or twice as far* from O, we shall find that the pull F' applied at N must be increased to 20 lb, or *twice as much force.* (3) If we wish to balance W acting at P by an *equal force,* we must apply such force at Q, 10 in. or *an equal distance* from O.

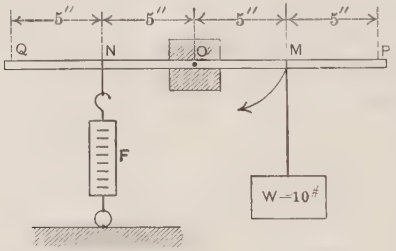


Fig. 63. Moments of force on a balanced rod.

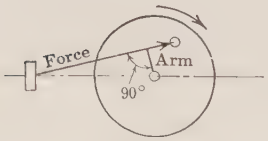


Fig. 64. Moment arm always at right angles with line of action of force.

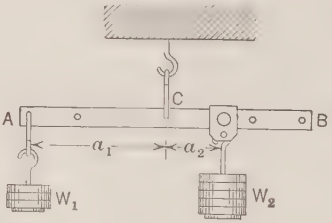


Fig. 65. Forces vertical, arms horizontal. $W_1 \times a_1 = W_2 \times a_2$.

In other words, we can balance a force by an *equal* force in an apparatus like this, if the two are applied at points equally distant from the pin; but *twice the force* is required to balance a force applied *twice as far* from the axis.

Two moments are thus equal when the product, *force times perpendicular distance from axis to force,* is the same for both. The measure of the moment of a force is, therefore, always the product obtained by multiplying the amount of the force by

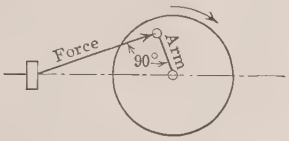


Fig. 66. Maximum moment. Arm = radius of crank circle.

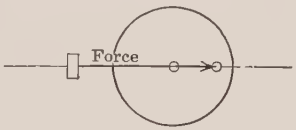


Fig. 67. Minimum moment. Arm = zero. "Dead center."

the length of the perpendicular from the action line of the force to the axis of rotation, or expressed as an equation,

Moment of force = force × moment arm

It must be kept in mind that the **moment arm is always perpendicular to the action line of the force,** as *a*, Fig. 61.

The thrust of the connecting rod tends both to turn the wheel and to push it against the axle. The rotational effect is measured by the product *force along the rod times a*, not by force times distance $P'X$. The maximum moment or torque will occur when the crank pin P is in such a position that the crank radius PX is at right angles to the connecting rod; then the arm a equals PX although in every other case it is less than PX, diminishing to zero when on "dead center," as indicated by the unbroken lines in Fig. 61. Obviously, any force along the rod times a zero arm will give a zero moment. See Figs. 64 to 67.

Fig. 68. Turning up a set-screw.

The torque of a motor is measured by the force on the belt times the radius of the motor pulley. The same torque can be obtained on a larger pulley with less force on the belt.

With a given force, a greater moment may be obtained by increasing the moment arm. Mechanics often slip a piece of pipe

over a wrench handle to increase its length. Brake drums on automobiles are made as large as conditions permit to give the force of friction provided by the brake band as long a moment arm as possible. Machinists frequently place a wrench on the blade of a screw driver, as in Fig. 68, to secure a longer moment arm and thus to enable them to apply more force when turning up a set screw.

PROBLEMS

5–1. A force of 1000 lb acts along a connecting rod on a crank pin 19 in. from the center of a flywheel. Find the maximum moment or torque. Find also the minimum moment.

5–2. If the pull on the belt in Fig. 13 is 40 lb and the diameter of the pulley is 14 in., what is the torque in pound-inches?

5–3. What tension in a belt would be caused by a torque of 2 lb-ft on a 4-in. diameter motor pulley?

Fig. 69. Making up a pipe joint.

5–4. Find the torque on a 6-in. diameter pulley due to a belt tension of 55 lb.

5–5. If a force of 50 lb is applied 4 ft from the fulcrum of a lever, what will be the moment in pound-feet?

5–6. What force applied 42 in. from the fulcrum of a lever will give a moment of 385 lb-ft?

5–7. The straight bar AB in Fig. 70 is pinned at A. Force F_1 is 40 lb, and force F_2 is 50 lb. Find the moment of each force about the pin. Which way will the bar rotate?

5–8. Figure 69 shows a pipe wrench being used to turn a section of pipe into a tee. What is the moment of a pull of 80 lb applied perpendicular to the wrench handle at a distance of 16 in. from the axis of the pipe?

5–9. Find the moment on the blade of the screw driver in Fig. 68 due to a pull of 50 lb at right angle to the wrench handle 14 in. from the axis of the screw.

5–10. In the bent arm lever of Fig. 42, arm AB is 11 in. and CB is 7 in. If a pull F_1 of 70 lb is put at A, determine its moment about B. What force F_2 at C would cause a moment of the same amount about B?

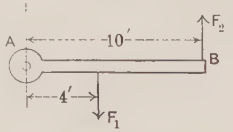


Fig. 70.

34. Conditions of equilibrium for parallel forces. The two conditions of equilibrium are that all the forces acting together

shall have no tendency to produce motion of translation, and no tendency to produce rotation about any chosen axis. As stated in the preceding article, the tendency of a force to cause rotation is measured by its moment. Therefore, **if the moments of all the forces about any axis of rotation are determined and are found to balance, there will be no tendency toward rotation.** This is the *law of moments.*

Suppose the rod AB, Fig. 71, to be acted upon by forces F_1, F_2, F_3, etc., which are parallel. It is evident that the effect of these forces may be any of the following: (1) to displace AB either upward or downward; (2) to cause AB to rotate about some point; or, (3) to produce both displacement and rotation at the same time.

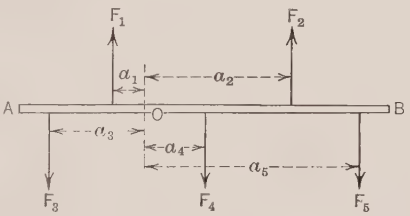


Fig. 71. Parallel forces in equilibrium.

Therefore, in order that AB may be in equilibrium, the tendencies toward displacement and rotation must each be so balanced as to have a total resultant of zero.

There will be no displacement of AB either upward or downward when the forces up equal the forces down; i.e., when $F_1 + F_2 = F_3 + F_4 + F_5$. Or, if the convention of the preceding chapters be used, that of considering forces up as positive and forces down as negative, then

$$+F_1 + F_2 - F_3 - F_4 - F_5 = 0$$

That is, *there will be no motion of translation when the algebraic sum of all the forces equals zero.*

To determine whether there will be any rotation, the perpendicular distance from each force along the bar to *any* selected point along the bar AB, as O, may be measured and the moments of the forces with respect to this point found. It is evident that forces F_1, F_4, and F_5 tend to produce rotation clockwise about O, and forces F_2 and F_3 tend to produce motion counter-clockwise. When these tendencies are equal, i.e., when

$$F_1 \times a_1 + F_4 \times a_4 + F_5 \times a_5 = F_2 \times a_2 + F_3 \times a_3$$

there will be no rotation about O. Or, letting clockwise moments be considered positive and counter-clockwise moments negative,

$$+F_1 \times a_1 - F_2 \times a_2 - F_3 \times a_3 + F_4 \times a_4 + F_5 \times a = 0$$

That is, *there can be no rotation about any selected point when the algebraic sum of all the moments equals zero.* This will be true if any point other than O is selected.

The conditions of equilibrium for parallel force systems are therefore fulfilled when:

(1) **The algebraic sum of all forces = 0**
(2) **The algebraic sum of the moments of all forces about any point = 0**

Stated as equations, the conditions of equilibrium for parallel forces are:

$$\Sigma V = 0 \tag{2}$$

$$\Sigma M = 0 \tag{3}$$

in which M represents the moment of a force, and ΣM the algebraic sum of the moments of all forces in the system, and ΣV represents the algebraic sum of all vertical forces (as in Art. 31).

If the parallel force lines were horizontal instead of vertical, equation (1) $\Sigma H = 0$, would be used instead of equation (2).

If the lines were neither horizontal nor vertical, the plane in which they act could be considered rotated until either the H or V axis was parallel with their direction.

35. Three parallel forces. Levers. Equilibrium with three parallel forces is a very common and important case in mechanics. A lever offers one example of three parallel forces.

In its simplest form a lever is a straight horizontal bar, Fig. 72, resting on a fulcrum O, with a load W at one end, and a force F at the other. F acts to lift the load, or to hold it in equilibrium. The moment of F about O equals F times FO; the moment of W about the same point equals W times the arm WO. Since the lever is in equilibrium, F times FO equals W times WO, from which the force to lift a given load can be found; also the load which can be lifted by a given force.

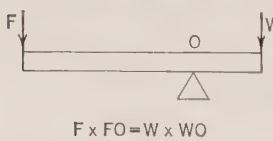


Fig. 72.

Levers are sometimes divided into three classes: the *first*, as above, with the fulcrum between the force and the weight; the *second*, with the weight between the force and the fulcrum; the *third* with the force between the weight and the fulcrum. In each lever the axis of rotation is at the fulcrum. Each case can be solved

by the method of moments, remembering always that the moment arm is the distance from the force or weight *to the fulcrum.*

The weighing beam of a platform scales, and the steel yard as used for weighing grain or cotton, are examples of the lever. Fig. 85 illustrates a lever, with the fulcrum at the end, A.

36. The resultant and equilibrant of parallel forces. The **resultant** of parallel forces will equal their algebraic sum. It will also equal the equilibrant in amount and will act at the same point but in the opposite direction. For example, Fig. 73 represents a parallel force system in which F_1 and F_2 act in the same direction on a weightless rod supporting a third force F_3. The resultant of F_1 and F_2 (that is, the force which could be substituted with the same result or effect) would equal $F_1 + F_2$, would act in the same direction (up), and would act at the same point as does F_3.

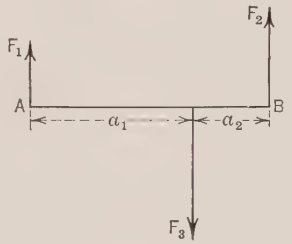


FIG. 73.

In general, the **equilibrant** for any number of parallel forces may be found by selecting a force of an amount to satisfy the $\sum V$ equation, and locating it at such a distance from any center of moments as to satisfy the $\sum M$ equation. For example, in the force system in Fig. 73, if F_1 and F_2 are acting on the weightless rod and it is required to find a third force, F_3, to produce equilibrium, the equilibrant $= F_3 = F_1 + F_2$. Further, by taking moments about any point, as A, the distance that the equilibrant is from A can be found from the equation of moments *about* A as an axis of rotation. The equation is:

$$F_3 \times a_1 = F_2 \times (a_1 + a_2)$$

from which the distance is

$$a_1 = \frac{F_2 \times (a_1 + a_2)}{F_3}$$

For this *special case,* with only three forces, the center of rotation could be taken at the point of application of F_3.

Then $$(F_1 \times a_1) + (F_3 \times 0) - (F_2 \times a_2) = 0$$

From which $$F_1 \times a_1 = F_2 \times a_2$$

And $$\frac{a_1}{a_2} = \frac{F_2}{F_1}$$

That is, the location of the equilibrant must be such that the arms will be in inverse proportion to the forces.

If *two* forces in a parallel system are *unknown in amount* they can be determined, provided their points of application are known, by selecting the center of moments at the point of application of one of them. That one will then have no moment (its moment arm equals zero), which leaves only one unknown term in the equation.

A simple beam, or horizontal member in a structure or a machine, supported at both ends, is an example of parallel forces. The supports furnish the two reactions to the one or more loads carried by the beam. Included as a load must be the *weight of the beam*, which can be considered as a single vertical force acting downward from the center of gravity of the beam. As stated in Art. 9, the center of gravity of a uniform beam is at its center point.

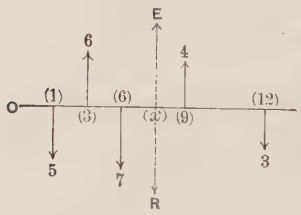


FIG. 74.

Example 1.

Find the amount, direction, and point of application of the resultant of the system of parallel forces shown in Fig. 74.

Solution. Find first the force E, the equilibrant. A force equal to this force, applied at the same point but in the opposite direction is the required resultant. The equation of forces gives

$$\text{for } \Sigma V: \quad 6 + 4 + E - 5 - 7 - 3 = 0$$
$$E = 5 + 7 + 3 - 6 - 4 = +5$$

Therefore, the equilibrant is an upward force of 5. And hence the resultant is a downward force of 5.

Taking moments about O, and letting x equal distance of E from O gives:

$$\text{for } \Sigma M: \quad (5 \times 1) - (6 \times 3) + (7 \times 6) - 5x - (4 \times 9) + (3 \times 12) = 0$$
$$5x = 5 + 42 + 36 - 18 - 36$$
$$5x = 29$$
$$x = 5.8$$

The resultant is thus found to be a force 5 acting down at a point 5.8 from O. *Ans.*

Example 2.

Find the force on each of the two supports of the beam shown in Fig. 75. The *force* diagram is below the *apparatus* diagram.

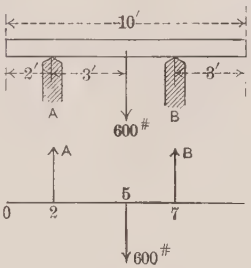


Fig. 75.

Solution. The equation of forces is,

$$\Sigma V: \quad A + B - 600 = 0$$

or

$$A + B = 600 \text{ lb}$$

For the equation of moments it is best to take moments about the point of application of one of the supports, *since in that case the moment of its reaction is zero*, and the unknown value of that reaction does not appear in the moment equation. Taking moments about the point of application of A gives:

$$\Sigma M: \quad 3 \times 600 - 5\,B = 0$$

$$5\,B = 1800$$

$$B = 360 \text{ lb} \quad \textit{Ans.}$$

Then substituting in the first equation:

$$A + 360 - 600 = 0$$

$$A = 240 \text{ lb} \quad \textit{Ans.}$$

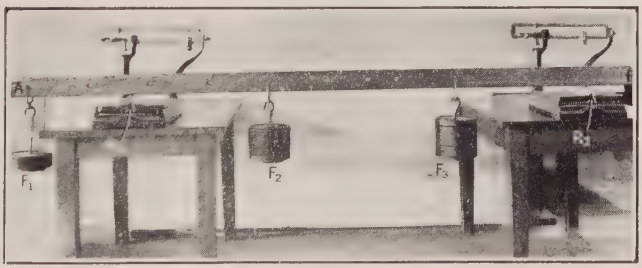


Fig. 76. Laboratory apparatus for parallel forces on a simple beam.

It may be noted here that the principle of moments can be applied, if so desired, to a system of concurrent forces.

Example.

In Fig. 37, three forces act at the point B, namely the weight W, the pull of the cord BC, and the push of the stick AB. Let $W = 50$ lb, $AB = 24$ in., angle at $B = 40°$. Solve for the forces in BC and AB.

Solution. Take the axis of rotation, or center of moments, at A. Then the arm for W equals 24 in., for the push of the stick equals 0, since its line of action passes through A, for the tension in the tie

equals the perpendicular distance from A to the line $BC = 24 \sin 40°$ $= 24 \times 0.643 = 15.4$ in. Call the tension in the stick T. Then,

$$\Sigma M: \quad (50 \times 24) - (T \times 15.4) = 0$$

$$T = \frac{1200}{15.4} = 77.9 \text{ lb} \quad \textit{Ans.}$$

To compute the force in the stick, take center of moments at C. Then the arm for $W = 24$ in. as before (measured to the line of action of W produced), the arm for the tie $= 0$, the arm for the stick $= 24$ tan $40° = 24 \times 0.839 = 20.1$ in. Call force in stick C. Then

$$\Sigma M: \quad (50 \times 24) - (C \times 20.1) = 0$$

$$C = \frac{1200}{20.1} = 59.7 \text{ lb} \quad \textit{Ans.}$$

37. Couples. Two equal and parallel forces acting in opposite directions, as in Fig. 77, are called a **couple.** The perpendicular distance a between the two forces is called the **arm** of the couple.

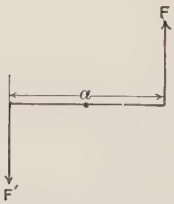


Fig. 77. A couple.

Since $F_1 = F_2$ it is evident from the laws which apply to parallel forces that:

1. A couple can produce rotation only.

The moment of a couple is equal to the product of the arm by one of the forces. The direction of rotation in Fig. 77 is counterclockwise.

2. A couple cannot be balanced by a single force.

In order to produce equilibrium a second couple having an equal moment is required.

The couple has many applications, such as the couple due to one's two hands on a steering wheel, the turning of a rowboat by pulling one oar and backing the other, and the banking of an airplane by raising one aileron and depressing the other.

Certain transformations may be made without changing the characteristics of a couple as regards magnitude and direction of rotation, which are useful in advanced problems in mechanics. Reference may be made to a text on engineering mechanics.

PROBLEMS

5–11. What force must be applied at the end of a lever 4 ft long to raise 550 lb at the other end if the fulcrum is 4 in. from the weight end? What load could be lifted with a force of 80 lb?

5–12. A lever is pivoted at one end with a weight of 91 lb $2\frac{1}{2}$ in. from that end and an upward force 30 in. beyond the weight. Find the force.

5–13. A lever 12 in. long has the fulcrum at one end, and a weight of 25 lb at the other. Find the amount of the force required to lift the weight, if it acts 8 in. from the fulcrum and 4 in. from the weight.

5–14. If the weight of the bar in Problem 5–11 is 25 lb, what force would then lift the 550-lb weight? (Where can the weight of the uniform bar be considered to act?)

5–15. A steel bar 4 ft long, weighing 11 lb, is suspended by a cord 8 in. from one end. What force will have to be applied at that end to balance the bar?

5–16. A steel bar weighing 10 lb, 24 in. long, is balanced over a knife edge 8 in. from the right end. What force must be applied at that end to balance the bar with a weight of 5 lb hung at the left end?

5–17. Find the force caused at B in each of the arrangements of the 18-in. rod in Fig. 78 if the force at A in each case is 10 lb.

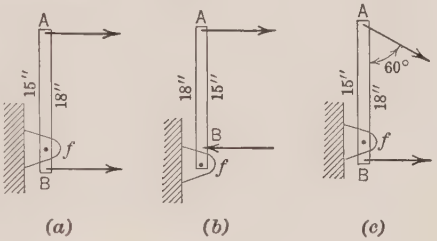


Fig. 78.

5–18. Given a rod 2 ft long and of negligible weight and a balance whose maximum capacity is 64 oz, show two ways in which you could arrange your apparatus to weigh with this balance a body weighing 14 lb. Give distances on your diagrams and proof of the correctness of your methods.

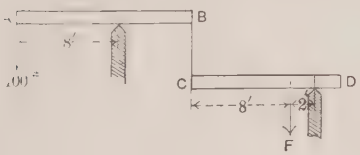


Fig. 79.

5–19. AB, Fig. 79, is a lever 14 ft long, and weighing 12 lb. CD is a lever 12 ft long and weighing 8 lb. The two are joined at the ends B and C by a flexible coupling. Assuming the weights of the levers to act at their middle points, what force can be exerted at F (8 ft from C) by a force of 100 lb applied at A?

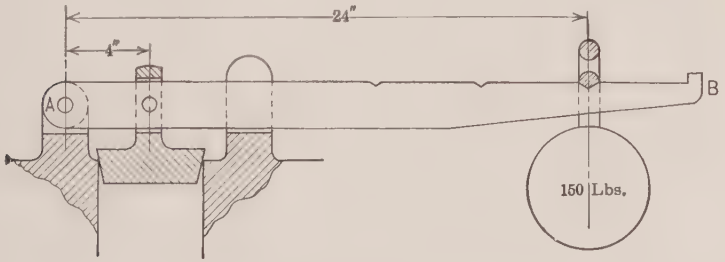


Fig. 80.

5–20. The safety valve lever AB, Fig. 80, weighing 12 lb, is pivoted at A. Its center of gravity is 10 in. from A. Weight of valve V is 8 lb, diameter of

valve 3 in. Valve is pivoted 4 in. from A. Find the pressure per square inch required to open the valve, when a weight W of 150 lb is suspended 2 ft from the end A of the lever.

5–21. A straight, uniform lever AB, 12 ft long, balances about a point 5 ft from B, when weights of 9 and 13 lb are suspended from A and B, respectively. Find the weight of the lever.

5–22. In the wheel and axle, Fig. 81, radius R is 10 in., radius r, 3 in. If P_1 is a force of 60 lb, what force must be applied at P_2 for equilibrium, disregarding friction? What is the pressure on the shaft at the center?

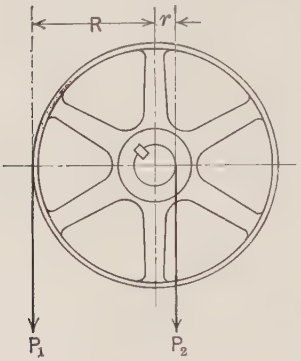


Fig. 81.

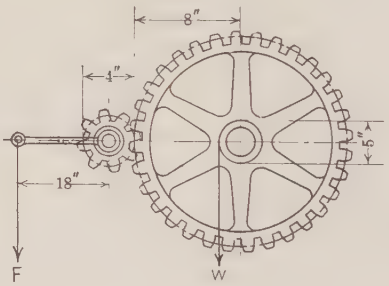


Fig. 82.

5–23. Figure 82 shows a common type of geared windlass. Dimensions are as indicated. If the force F is 180 lb and friction is negligible, what load will be supported at W? What force will be exerted on the shaft of the small gear?

5–24. A lever AB, 10 in. long, balanced on a fulcrum F, carries a weight of 500 lb at B and 50 lb at A. Find the amount of the resultant and the equilibrant, and distance from A to their point of action.

5–25. Find the amount and position of the resultant of 3 forces acting on a bar 30 in. long, if 10 lb acts 10 in. from the left end, 12 lb, 12 in. from the left end, and 14 lb, 8 in. from the right end.

5–26. A bar, AB, is 40 in. long and is acted upon by a force F_1, 5 lb down 5 in. from A; by F_2, 10 lb down, 10 in. from A; by F_3, 17½ lb down, 30 in. from A; and by F_4, 32½ lb up, 20 in. from A. Is the bar in equilibrium? Check your answer by taking moments about more than one point.

5–27. In the preceding problem (*a*), if instead of F_4 being 32½ lb at 20 in. from A, it were 20 lb at 32½ in. from A, would the bar be in equilibrium? (Other forces unchanged in amount or position.) Why? Again (*b*) if F_4 were 32½ lb at 21 in. from A, would the bar be in equilibrium? Why?

5–28. Determine whether or not the force system in Problem 2–4, Chapter II, is in equilibrium. Show proof of your answer.

5–29. A rod 6 ft long, the weight of which may be neglected, rests upon two supports placed under the ends. Where must a weight of 22 lb be hung in order that the force on one support may be 9 lb?

5–30. A steel beam, 14 ft long, supports a load of 1600 lb, 5 ft from the left end. Find the reactions of each support.

5–31. Find the amount of weight carried by each pier of a bridge of 28-ft span due to a truck weighing 5 tons standing: (*a*) at the middle, (*b*) 9 ft from the left end, (*c*) 3 ft from the left end.

5–32. Find the end reactions of a beam 16 ft long, weighing 500 lb and carrying a load of 1500 lb 4 ft from the left end, and another load of 1000 lb 6 ft from the left end.

5–33. A beam AB 18 ft long, weighing 16 lb per linear ft, is supported at the ends. It carries a load of 1000 lb 3 ft from B. Find the end reactions.

5–34. A beam AB, 12 ft long, weighing 15 lb per linear ft, is supported by two reactions, R_1 at A and R_2 2 ft from B. A load of 500 lb rests 4 ft from B. Find the reactions.

5–35. A beam AB weighing 50 lb, 12 ft long, is supported by columns 3 ft from each end. The loads are: F_1 100 lb 1 ft from A; F_2 200 lb 3 ft from F_1; F_3 300 lb 4 ft from F_2; F_4 400 lb 3 ft from F_3 and 1 ft from B. Find the reactions R_1 and R_2.

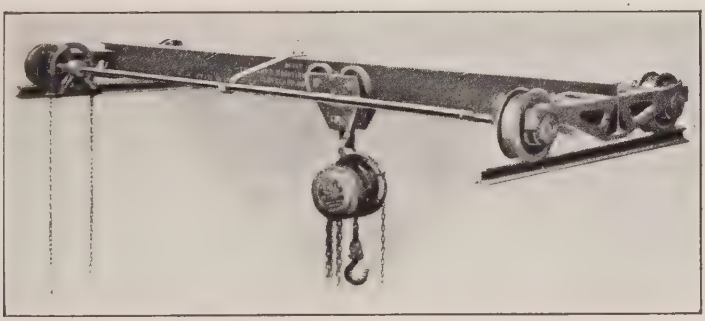


Fig. 83.

5–36. The weight of the single I-beam traveling crane shown in Fig. 83 is 3500 lb, and the span from the center of one rail of the track to the center of the other rail is 16 ft. The trolley carrying the chain hoist is 5 ft from the right-hand rail. The combined weight of the trolley and chain hoist is 800 lb. Find the force on each rail when a load of 2000 lb is carried by the hoist.

38. Center of gravity. The center of gravity of a body is the point at which the force of gravity may be considered always to act; that is, it is the point of application of the resultant of all those little parallel forces which gravity exerts on the particles of the body regardless of the way in which the body may be

placed or turned. As stated in Art. 9, the center of gravity of a uniform bar is on its central axis at the middle of its length; that of a cylinder at one-half its height; of a cone at one-fourth its height; the same for a pyramid; of a hemisphere at 0.375 times its radius from the center.

The position of the center of gravity of various-shaped bodies often must be determined in order to use engineering formulas for the design or analysis of structural or machine parts. One such formula is that involving the neutral axis of the cross section of a beam, or other member subjected to bending. The neutral axis passes through the center of gravity of the cross section of the beam. See Art. 130.

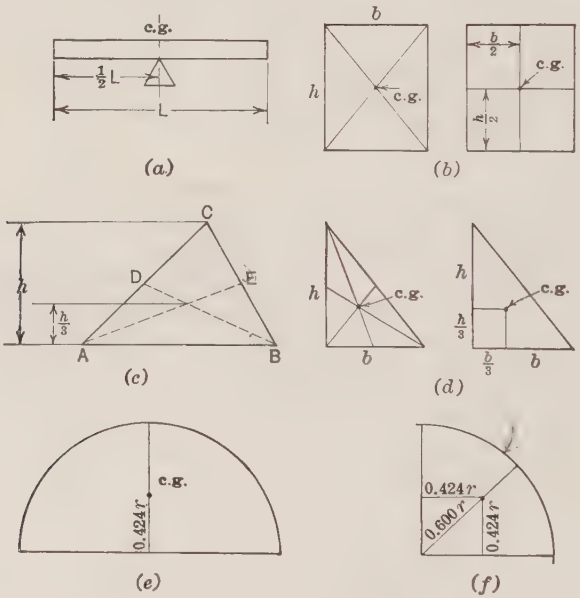


FIG. 84. Center of gravity of geometrical areas.

For cross-sectional areas of uniform shape the location of the center of gravity is a simple matter of geometry. Thus the center of gravity of a square or *rectangular surface* is at the intersection of its diagonals which is at one-half the height and one-half the breadth; of a *triangle* at the intersection of the medians, which is at a point one-third the altitude above the base; of a *circle* at the center, the intersection of two diameters; of a *semi-circle* at a

point on the axis of symmetry at a distance 0.42 times the radius from the center; of a *quadrant* or quarter circle at a distance 0.42 times the radius from each bounding radius, also on the axis of symmetry at a distance 0.60 times the radius from the center. See Fig. 84.

The center of gravity of other *regular figures* may be found from handbooks, and of *structural shapes* from the steel companies' handbooks.

The center of gravity of an *irregular shape*, or of a *combination* of regular shapes, can be computed by the law of moments. It can also be determined by a simple experimental method.

39. Experimental method of finding center of gravity. It was shown in Art. 22 that the center of gravity of a body, suspended from a single point in such a way that it can turn freely about the support, must always be at some point in the vertical line under the support.

An experimental method is to draw the shape to scale, cut it out of thin metal or cardboard, and suspend it swinging freely on a pin, from two or more different points in succession. Suspend a plumb-bob from the pin and draw the vertical line on the card for each case. The intersection of these lines is at the center of gravity.

Another method is to balance the figure on a knife edge, marking the location of the edge across the figure. Make two or more locations, the intersection of the lines giving the position of the center of gravity.

40. Center of gravity by moments. The computation of the position of the center of the gravity of an irregular body is as follows.

Example 1.

Suppose it is desired to find the center of gravity of a steel bar 32 in. long, 1 in. square for one-half its length, and $\frac{1}{2}$ in. square for the remainder of its length and symmetrical about its long axis.

Solution. The position could be determined experimentally by balancing the bar on a knife edge or fulcrum block. If this were done, the bar would be subject to a system of three parallel forces, the weight of the larger section acting down at the center of gravity of that section, the weight of the smaller section acting down at its center of gravity, and the reaction of the fulcrum equal to the total weight acting up at the center of gravity of the whole body. Taking moments about one end of the bar, say the larger end, the distance x from that

end can be determined. The weights are proportional to the volumes, since the material is homogeneous; therefore volumes can be used in place of forces in the equation.

The volume of the large end $= 1 \times 1 \times 16 = 16$ cu. in.; its arm is 8 in.

The volume of the small end $= \frac{1}{2} \times \frac{1}{2} \times 16 = 4$ cu. in.; its arm is 24 in.

The total volume $= 16 + 4 = 20$ cu. in.

The equation of moments about the larger end is:

$$(16 \times 8) + (4 \times 24) - (16 + 4)\, x = 0$$

$x = \dfrac{128 + 96}{20} = 11.2$ in. the distance of the center of gravity from the large end. *Ans.*

If the bar were not symmetrical about its length, an equation could be set up about an axis parallel to its length, as at one edge, and a distance y, at right angles to x, determined.

Example 2.

Again suppose that a hole of $\frac{1}{2}$ in. diameter, 6 in. long, is drilled into the large end of the above bar, along its axis.

Solution. This air space can be treated as an upward force to counteract the weight of the steel taken out. Its volume $= 6 \times (\frac{1}{2})^2 \times 0.7854 = 1.18$ cu. in.; its arm $= 3$ in. The equation now is:

$$(16 \times 8) + (4 \times 24) - (1.18 \times 3) - (20 - 1.18)\, x = 0$$

$$x = \frac{128 + 96 - 3.54}{18.82} = 11.7 \text{ in.} \quad \textit{Ans.}$$

Example 3.

Suppose again that the 6-in. hole is filled with lead.

Solution. Because the material is not homogeneous the equation must now include the weight instead of volume, which for steel is 0.28 lb per cu in., and for lead is 0.41 lb per cu in.

The gross weight of the large end $= 16 \times 0.28 = 4.48$ lb

The weight of the small end $= 4 \times 0.28 = 1.12$ lb

The weight of steel removed $= 1.18 \times 0.28 = 0.33$ lb

The weight of the lead $= 1.18 \times 0.41 = 0.48$ lb

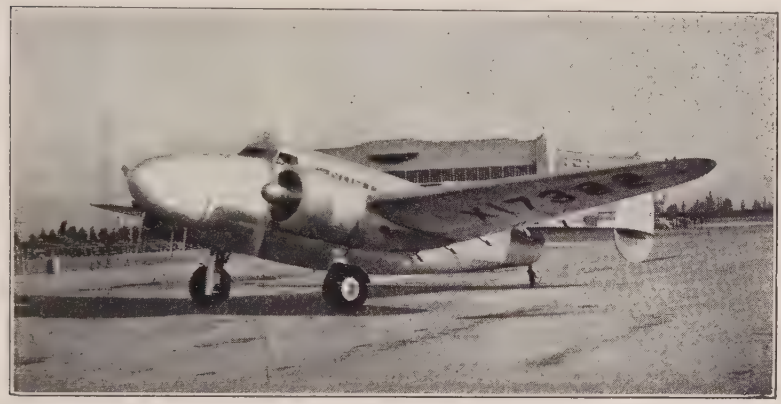
The total weight of the body $= 4.48 + 1.12 + 0.48 - 0.33 = 5.75$ lb

The equation now is:

$$(4.48 \times 8) + (1.12 \times 24) - (0.33 \times 3) + (0.48 \times 3) - 5.75\, x = 0$$

$$x = 11.0 \text{ in.} \quad \textit{Ans.}$$

The center of gravity therefore was moved about $\frac{1}{2}$ in. by drilling the hole, and was moved back about $\frac{3}{4}$ in. by the lead plug.



Courtesy Lockheed Aircraft Corporation.

FIG. 85(*a*). Twin-motored transport airplane.

As a further illustration of the application of the method of moments, the center of gravity of such bodies as automobiles, cars, and airplanes is thus determined.

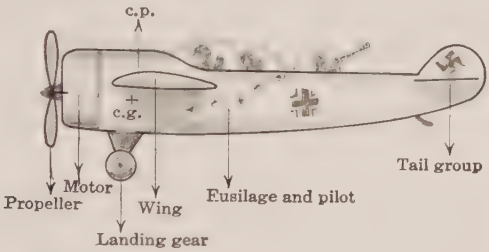


FIG. 85(*b*). Center of gravity of airplane.

Example 4.

Find the position of the center of gravity of an airplane, in horizontal position, measured along a horizontal axis back from the center line of the propeller; given the following data:

wt of propeller: 32 lb
wt of motor: 420 lb, cg 7 in. from cl of prop
wt of wing: 2 lb per sq ft area
size of wing: 5 ft $\times$ 30 ft, cg 6 ft 3 in. from cl of prop
wt of fusilage, including pilot: 500 lb, cg 8.5 ft from cl of prop
wt of tail piece: 48 lb, 20 ft from cl of prop
wt of landing gear: 50 lb, cg 3 ft 6 in. from cl of prop

Solution.

Total wt (by ΣV) $= 30 + 420 + 300 + 500 + 48 + 50 = 1350$

Take moments about the center line of propeller ($\Sigma M = 0$).

$$(32 \times 0) + (420 \times 1\tfrac{7}{12}) + (50 \times 3.5) + (300 \times 6.25) + (500 \times 8.5) + (48 \times 20) - 1350\,x = 0$$

$$x = 5.6 \text{ ft} \quad \textit{Ans.}$$

41. Center of gravity of areas. The computation of the position of the center of gravity of an area is similar to that of a solid.

Example 1.

Suppose it is desired to compute the position for the sectional area of Fig. 86.

Solution. Imagine this drawn to scale and cut out of a sheet of thin metal or card, and held in a horizontal position hinged along the axis XX' (for convenience at one edge of the card). Unless supported also at some other point the card will swing down about the hinge into a vertical position. The reason for its swinging down can be attributed to the moments due to the weight of each rectangle times its moment arm. Since weight can be considered to act at the center of gravity, the moment arm is the distance from the center of gravity to the hinge.

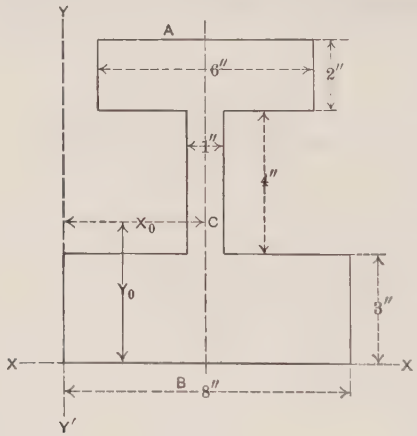


Fig. 86.

If the card is of uniform thickness and material, the weight of each of the three rectangles of which it is composed will be proportional to their areas, and the weight of the whole proportional to its total area. Since a force equal to its weight acting at the center of gravity of the card will support it, we may write as our equation of moments about the XX' axis,

$$(24 \times 1.5) + (4 \times 5) + (12 \times 8) - (40 \times Y_0) = 0$$

$$Y_0 = 3.8 \text{ in.}$$

Or, the center of gravity of the card is somewhere on a line parallel to axis XX' and 3.8 in. from it.

From the symmetry of the card, we know that the center of gravity will be located on the center line AB. Hence the center of gravity of the card is at a point C on AB, 3.8 in. from B.

If the figure were not symmetrical, we could take moments about an axis YY' at right angles to XX', and determine the distance X_0 from this axis to the center of gravity C.

Thus for Fig. 86,

$$(24 \times 4) + (4 \times 4) + (12 \times 4) - (40 \times X_0) = 0$$

$$X_0 = 4 \text{ in.}$$

The center of gravity C is thus at a point 3.8 in. from XX' and 4 in. from YY'. *Ans.*

From the preceding it is evident that the center of gravity of any figure may be found from the general equations:

$$AX_0 = \Sigma(ax) \tag{1}$$

$$AY_0 = \Sigma(ay) \tag{2}$$

in which A is the total area of the figure, X_0 the distance from the center of gravity of the figure to an assumed axis YY', Y_0 the distance from the center of gravity to an axis XX' at right angles to YY', and $\Sigma\ (ax)$ and $\Sigma\ (ay)$ the sum of the products of each portion of the figure multiplied by the distance of its center of gravity from the Y and X axes respectively.

The student should test this general method by computing the center of gravity of various thin laminas cut from cardboard and then comparing the computed result with the experimental result as in the preceding article.

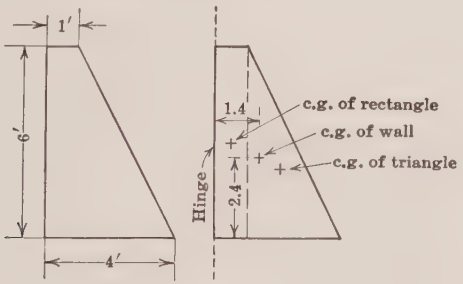


FIG. 87.

Example 2.

Consider the cross section of a retaining wall 6 ft high, 1 ft wide on top, 4 ft wide at the base, and with the left side vertical. See Fig. 87. The shape includes a rectangle and a triangle. See Fig. 84 for the center of gravity of the triangle.

Solution. Taking the XX' axis at the bottom edge, the equation is:

$$(6 \times 3) + (9 \times 2) - (15 \times Y) = 0$$
$$Y = \tfrac{36}{15} = 2.4$$

With the YY' axis at the left edge:

$$(6 \times 0.5) + (9 \times 2) - (15 \times X) = 0$$
$$X = \tfrac{21}{15} = 1.4$$

Thus the center of gravity of this wall is at a point 2.4 ft above the base, and 1.4 ft from the left or vertical side. *Ans.*

A second solution of the foregoing area, which illustrates the use of *negative* areas and moments, is as follows. In some areas, particularly with circular arcs, this may be necessary or at least provides the simplest procedure. Assume the shape to consist of an over-all rectangle, 6 by 4, from which a triangle 6 by 3 is to be deducted.

Then $(24 \times 3) - (9 \times 4) - (15 \times Y) = 0$

$Y = (72 - 36) \div 15 = 2.4$ as before

Also $(24 \times 2) - (9 \times 3) - (15 \times X) = 0$

$X = (48 - 27) \div 15 = 1.4$ as before

An interesting case is furnished by bodies bounded by curved lines, as Fig. 88. The area of the figure or its parts cannot be computed readily in this case. A result sufficiently good for practical use may be obtained, however, by dividing the figure into narrow elements of equal width by lines drawn parallel to the YY' axis. Then, if the elements are sufficiently narrow, the length of the central line of each, as a_3b_3 in the figure, will be very nearly the *average length of* the element, and this times the width will give the area. Then, if d be the width of each element,

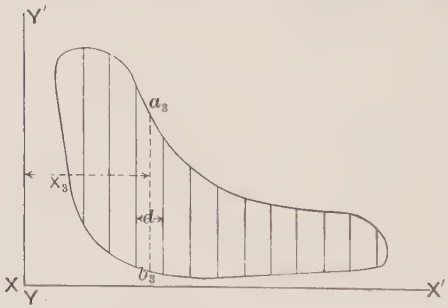


Fig. 88. Center of gravity of figure bounded by curved lines.

$$\Sigma(d \times a_1b_1 + d \times a_2b_2 + d \times a_3b_3 + \dots)$$
$$= \text{sum of areas of all the elements}$$
$$= \text{area of whole figure}$$

If x_1, x_2, etc., be the distances from the YY' axis to the central line of each element,

$\Sigma(d \times a_1b_1 \times x_1 + d \times a_2b_2 \times x_2 + \ldots)$ = sum of the moments of all the elements about the YY' axis.

Then, by the general equation above,

Area whole figure $\times X_0$ = sum of moments of all elements, or,

$$X_0 = \frac{\Sigma(d \times a_1b_1 \times x_1 + d \times a_2b_2 \times x_2 + \ldots)}{\Sigma(d \times a_1b_1 + d \times a_2b_2 + \ldots)}$$

By dividing the figure into elements parallel to the XX' axis, the value of Y_0 may be found in a similar manner.

The moment of inertia. When, in the design or analysis of a beam or other member subjected to bending stresses, the neutral axis of a section has been determined by the location of the center of gravity, there is another concept which is required, known as the **moment of inertia of the section or area.** The moment of inertia is represented by the expression $I = \Sigma ar^2$. This is considered in the chapter, Strength of Materials, in Art. 129, which may be taken up at this point if desired.

PROBLEMS

5–37. At what point may the weight of a 2 by 4 in. beam 10 ft long be considered to act?

5–38. A triangular surface measures 6 in. on each side. How far from one of the sides is its center of gravity?

5–39. The cross section of a retaining wall is a triangle with one side vertical, 4 ft 6 in. high, and 24 in. wide at the base. Find its center of gravity.

5–40. A section of masonry cornice is a quarter circle of 11-in. radius, with one radial side horizontal. How far from this side is the center of gravity?

5–41. A certain machine part is semi-circular with a 14-in. diameter and 1.25 in. thick. Find its center of gravity.

5–42. Compute the position of the center of gravity of an iron bolt having the following dimensions: Head, $\frac{7}{8}$ in. by $\frac{7}{8}$ in. and $\frac{7}{16}$ in. thick; threaded part, $\frac{7}{16}$ in. diameter and 3 in. long.

5–43. A round rod of steel, 26 in. long, weighing 18 lb, has two sliding weights clamped on it near one end. Both sliders are cylindrical, and symmetrical about the axis of the rod. One weighs 6 lb and its center is 3 in. from the left end; the other weighs 3 lb with its center 2 in. from the first. Compute the position of the center of gravity of the weighted rod.

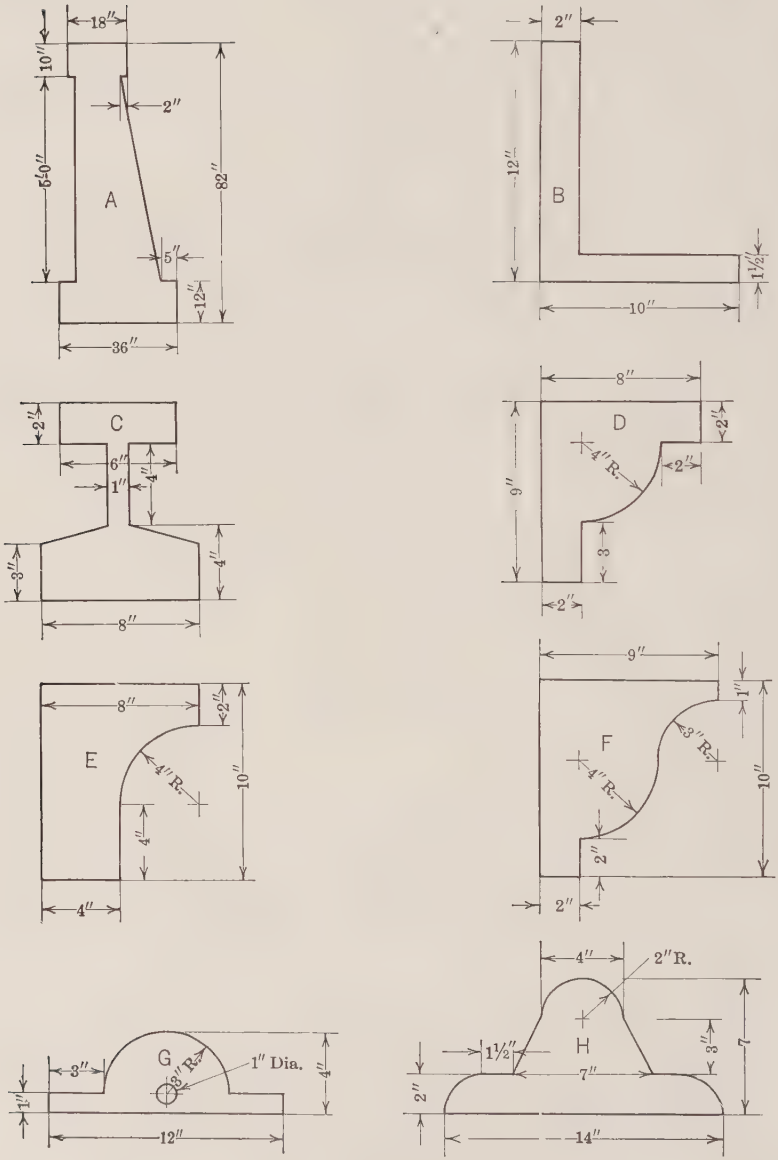

18″
10″
50″
2″
A
82″
5″
12″
36″
2″
12″
B
1½″
10″
2″
C
6″
1″
4″
8″
4″
8″
8″
D
4″ R.
2″
9″
3
2″
2″
8″
E
4″ R.
10″
4″
4″
9″
1″
F
3″ R.
4″ R.
10″
2″
2″
4″
2″ R.
H
1½″
3″
7″
7
2″
14″
G
3″ R.
1″ Dia.
4″
1″
3″
12″

Fig. 89.

5–44. Find the center of gravity of a steel rod 28 in. long, of 1-in. diameter for one-third its length and $\frac{1}{2}$-in. diameter the remainder, symmetrical on its long axis.

5–45. Find the center of gravity of the bar in Example 3, Art. **40**, if the hole is filled with brass weighing 0.31 lb per cu in., instead of with lead, and if a length of 8 in. is cut off the $\frac{1}{2}$-in. diameter end.

5–46. Find the center of gravity of a circular cast-iron plate, 7 in. in diameter, with a hole 1 in. in diameter half way between the center and the edge.

5–47. Compute the position of the center of gravity of the cross-sectional area of a wall, 9 ft high, 2 ft wide on top, 5 ft wide at the base, with one side vertical.

5–48. Same for a $\perp$ beam with an over-all height of 10 in., width 8 in. and thickness of both vertical and horizontal portions of $2\frac{1}{4}$ in.

5–49. Same for a wall symmetrical about a vertical axis with following dimensions: Coping, 18 in. wide by 10 in. high; wall proper, 14 in. wide under coping; 26 in. wide at top of footing, 5 ft high; footing, 3 ft wide by 1 ft high.

5–50. Compute the position of the center of gravity of an area in Fig. 89.

SUMMARY

The moment of a force is its tendency to cause rotation about a given axis. It is equal to the force times its moment arm. The arm is the perpendicular distance from the line of action of the force to center or axis of rotation.

Torque is the term for moment used with machinery and motors.

The law of moments is that if the moments of all forces about any axis of rotation balance, there will be no tendency toward rotation. That is true if the sum of the clockwise moments minus the sum of the counterclockwise equal zero.

The conditions of equilibrium for parallel forces are expressed by the equations:

$$\Sigma V = 0 \text{ (for motion of translation)} \quad (2)$$

$$\Sigma M = 0 \text{ (for motion of rotation)} \quad (3)$$

If the force system is horizontal, $\Sigma H = 0$, equation (1), is substituted for equation (2).

The **resultant** of parallel forces is found by first finding the equilibrant, which is equal and opposite to it.

The **equilibrant** is found by using the ΣV equation to determine the amount, and then using the ΣM equation to determine the position at which it acts. In this procedure in the moment equation the forces will all be known quantities, the unknown term being the

arm of the equilibrant, that is, the distance from the axis of rotation to the point at which it acts. The axis may be chosen at *any* point, quite often for convenience at the end of the bar or beam.

Two unknown forces, provided their positions are known, as in the supporting reactions of a beam, can be found by using the moment equation with the axis at the point of action of one of the unknowns. This force will then have no moment and solution can be made for the other.

Levers are a special case of parallel forces. The axis of rotation is at the fulcrum, and the moment arm must be measured to the fulcrum.

A couple is caused by two equal parallel forces acting in opposite directions. It is equal to the arm between the forces times one force. It can produce rotation only, and cannot be balanced by a single force.

The center of gravity of a body is the point at which its weight can be considered to act. The centers of gravity of some regular shapes are given in Art. 38. The center of gravity of irregular bodies and areas can be found by the moment equation. The procedure is to set up an equation in which the sum of the moments of each regular shape composing the whole body equals the moment of the whole acting about the same axis. For convenience one axis is usually chosen at the bottom and except when the figure is symmetrical, another at the left edge. See Art. 40 and 41.

The **moment of inertia** is an advanced application of the principle of moments, and is treated in the chapter, Strength of Materials.

CHAPTER VI

NON-CONCURRENT FORCES

This chapter treats of the third group of force systems in one plane. Non-concurrent forces are those whose lines of action are not all parallel and do not all pass through a common point. The conditions for equilibrium of this group will be discussed, together with the method of solving for the unknown forces. The principle of free bodies also will be considered as well as the use of free body diagrams in the solution of problems.

42. Conditions of equilibrium for non-concurrent forces. In general, for any system of forces, equilibrium requires that there shall be no unbalanced force to produce change in translation, and no unbalanced moment to produce change in rotation. Equations (1) and (2) of the preceding chapters express these conditions as regards translation, and equation (3) as regards rotation.

For concurrent forces it is necessary to use both equations (1) and (2), after resolving all forces into components along horizontal and vertical axes. For parallel forces it is necessary to use only one equation for translation, together with the equation for rotation. For non-concurrent forces, since they are not all parallel, it is necessary to resolve them into H and V components and use both equations for translation; and since their lines of action do not all meet at a common point it is also necessary to use the equation for rotation.

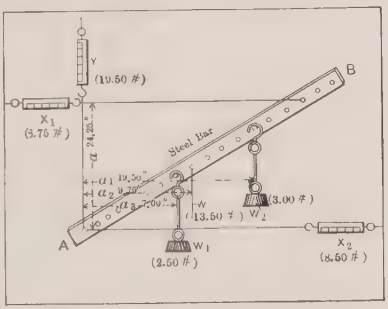


FIG. 90. Laboratory apparatus for non-concurrent forces.

Figure 90 represents a heavy bar AB acted upon by non-concurrent forces. A little consideration of the apparatus will show that if the forces on the bar are to be in equilibrium, i.e., if the bar is to remain at rest:

1. Force X_2 to the right must equal force X_1 to the left.

2. Force Y upward must equal the sum of W, W_1, and W_2, the forces downward.

3. Any moment of force tending to turn the bar counterclockwise about a selected axis must be balanced by an equal moment with respect to the same axis, tending to turn the bar clockwise. Let the axis be at A.

The three equations for this apparatus will be:

$$\Sigma H: \quad X_2 - X_1 = 0 \tag{1}$$

$$\Sigma V: \quad Y - (W + W_1 + W_2) = 0 \tag{2}$$

$$\Sigma M: \quad W \times a_2 + W_1 \times a_3 + W_2 \times a_1 - X_1 \times a = 0 \tag{3}$$

It will be noted that the moments of the forces X_2 and Y are omitted from equation (3), as each force has a moment arm of zero with respect to the chosen axis.

Figure 91 illustrates the *general case*, the heavy irregular line representing *any* body acted on by non-concurrent forces. It is evident that the forces F_1, F_2, F_3, and F_4 may be resolved into horizontal and vertical components, and that if

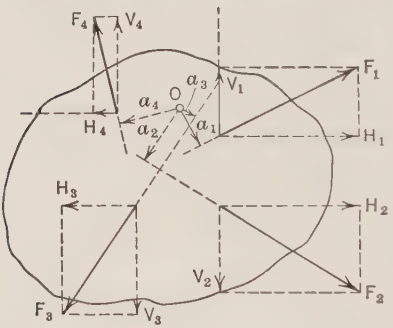


FIG. 91. Non-concurrent forces in equilibrium.

$$H_1 + H_2 - H_3 - H_4 = 0$$

there will be no tendency for displacement in a horizontal direction; if

$$V_1 + V_4 - V_2 - V_3 = 0$$

there will be no tendency for displacement in a vertical direction.

Also the perpendicular distance from each force line to *any* point O in the plane of the forces may be determined and the moments of the forces with respect to an axis through this point computed. If the clockwise moments equal the counterclockwise moments, or, in other words, if

$$(F_3 \times a_3) + (F_4 \times a_4) - (F_1 \times a_1) - (F_2 \times a_2) = 0$$

there will be no tendency toward rotation about O.

The conditions for equilibrium of non-concurrent forces can be stated thus:

A body acted upon by a system of non-concurrent forces in one plane will be in equilibrium when the sum of all horizontal components equals zero, the sum of all vertical components equals zero, and the sum of all moments about any axis equals zero.

Stated in symbols this is:

$$\Sigma H = 0 \tag{1}$$

$$\Sigma V = 0 \tag{2}$$

$$\Sigma M = 0 \tag{3}$$

43. Free bodies. As was pointed out in Chapter I, a force acting on a body is always exerted *by another body.* The force action between two bodies always consists of *two* equal and opposite *forces*; one force is exerted on body A by body B, the other force, equal in amount but opposite in direction, is exerted on body B by body A. It is always important in attempting the solution of any given problem, to avoid confused thinking by first selecting the body to which the forces are applied and by considering only the *forces acting on that body.*

A body thought of as removed from contact with all other external bodies and acted on by forces only is spoken of as a **free body.** Any structure as a whole or any member or part of a structure may be taken as the free body. A diagram showing the direction and point of application of the forces on the chosen body should then be made. This diagram should show *all forces*, including all applied loads and all forces exerted by any parts of the structure assumed to have been removed.

Thus we may consider the I-beam of the traveling crane, Fig. 83, as the "free body," in which case we shall have the force diagram of Fig. 92 in which R_1 and R_2 are the upward *pushes* of the rails, W the weight of the crane, that is, the gravity *pull* of the earth, and F the applied load.

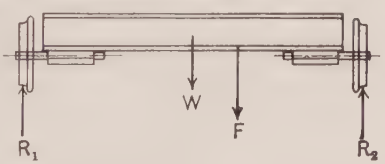


FIG. 92. Traveling crane represented as a free body.

With some exceptions, it is simpler to follow the convention of representing the forces as *pulls*, from the body. Thus, in Fig. 75 the free body diagram of the beam is shown with the two reactions as upward pulls.

The pulley at L of the derrick, Fig. 60, considered as a free body, gives the force diagram of Fig. 93 in which F_2 and F_3 represent the tension in each part of the hoisting rope (neglecting the friction of the pulley block this would be the same on both sides), and F_1 the force on the hook due to the load being lifted.

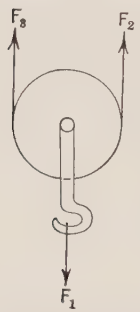


Fig. 93. Pulley of derrick as a free body.

The jib or boom CD of the same derrick, considered as a free body, gives the force diagram of Fig. 94(a). T indicates the force applied by the pull in the tie DE, F_1 the force due to the load at D, R the reaction at C, W the weight of the jib, and F_2 the tension in the hoisting rope running from D to the drum of the hoisting engine. (This rope is not shown in the figure.) If we neglect friction, F_2 is evidently equal to $\frac{1}{2}F_1$.

In many cases the exact direction of the reaction R may not be known. It is evident, however, that the hinge at C, Fig. 60, keeps the end of the jib from dropping down or being pushed toward the right. These two effects may be shown in

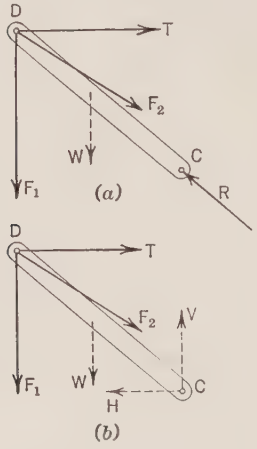


Fig. 94. Jib of derrick as a free body.

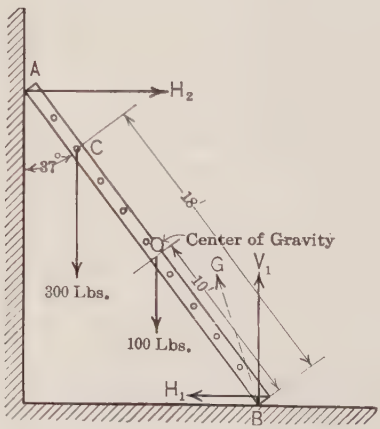


Fig. 95. Forces on a ladder.

place of R, as V and H of Fig. 94(b), and this diagram used instead of that of Fig. 94(a).

Figure 95 illustrates the free body diagram of a ladder AB (except that the shaded lines representing building and ground could

be omitted). The reaction of the building on the ladder is shown as H_2, the reaction of the ground as G. The ground reaction is replaced by its components H_1, which resists sliding, and V_1.

44. Solutions for typical systems of non-concurrent forces. The method of solution for practical systems of non-concurrent forces will be understood from the following typical examples:

Example 1.

Figure 95 represents the common case of a ladder leaning against a wall. Assume the ladder to be 24 ft long and to weigh 100 lb, its center of gravity being 10 ft from the end B. Assume further that the ladder, carrying a load F of 300 lb applied at C, 18 ft from the end B, makes an angle of 37° with the wall and that the friction between the end A and the wall is negligible. It is required to find V_1 and H_1, the vertical and horizontal ground reactions at B; the amount and direction of G, which represents the total force exerted by the ground; and the horizontal wall reaction H_2 at the end A.

Solution. From the requirements $\Sigma H = 0$, $\Sigma V = 0$, and $\Sigma M = 0$, we have the equations,

$$\Sigma H: \quad H_2 - H_1 = 0 \tag{1}$$

$$\Sigma V: \quad V_1 - 100 - 300 = 0 \tag{2}$$

$$\Sigma M \text{ (about } B): \quad H_2 \times 24 \cos 37° - 100 \times 10 \sin 37° - 300 \times 18 \sin 37° = 0 \tag{3}$$

Solving equation (3)

$$19.2\, H_2 - 602 - 3250 = 0$$

$$19.2\, H_2 = 3852 \text{ lb}$$

$$H_2 = 201 \text{ lb} \quad \textit{Ans.}$$

Substituting this value in equation (1),

$$H_1 = H_2 = 201 \text{ lb} \quad \textit{Ans.}$$

or the friction, etc., to keep the ladder from slipping is 201 lb.

From equation (2), V_1, the vertical component of the ground reaction at $V = 400$ lb. *Ans.*

The *total* ground reaction at B is, therefore,

$$\sqrt{(201)^2 + (400)^2} = 447 \text{ lb} \quad \textit{Ans.}$$

The 447-lb force acts in the direction G, making an angle with the ground whose tangent is $\frac{400}{201} = 2.00$.

The angle is, therefore, approximately 63°. *Ans.*

The ladder makes an angle of 53° with the ground; therefore it may be noted that the force exerted upon the ground by the ladder is not a thrust directly along the ladder in the direction AB.

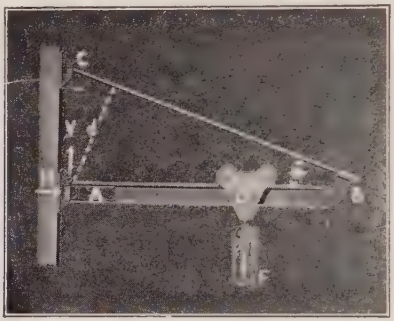


Fig. 96. Wall crane.

Example 2.

In the wall crane, Fig. 96, suppose $AB = 6$ ft, $AC = 3$ ft, and that the load F is applied 2 ft from B.

Solution. $CB = \sqrt{36 + 9} = 6.7$ ft, approximately. Therefore, $\cos\theta = \frac{6}{6.7}$, $\sin\theta = \frac{3}{6.7}$, and, since $\frac{d}{6}$ also $= \sin\theta$, $d = 6\sin\theta$.

The bar AB is held in equilibrium by the tension T in BC, the reactions V and H at the wall and the load F. Therefore,

$$\Sigma H: \quad H - T\cos\theta = 0 \tag{1}$$

$$\Sigma V: \quad V + T\sin\theta - F = 0 \tag{2}$$

$$\Sigma M: \quad F \times 4 - T \times 6\sin\theta = 0 \tag{3}$$

If the load F is known, T may be determined from equation (3). By substituting the value for T in equations (1) and (2), the values of the hinge reactions at the wall may be found. Suppose $F = 100$ lb. Then $100 \times 4 = T \times 6 \times \sin\theta$ (3)

$$T = \frac{400}{6 \times 3/6.7} = \frac{400 \times 6.7}{18} = 149 \text{ lb} \quad \textit{Ans.}$$

$$H = 149 \times \cos\theta = 149 \times \frac{6}{6.7} = 133\tfrac{1}{3} \text{ lb} \quad \textit{Ans.} \tag{1}$$

$$V = 100 - 149 \times \sin\theta$$

$$= 100 - 149 \times \frac{3}{6.7} = 100 - 66\tfrac{2}{3} = 33\tfrac{1}{3} \text{ lb} \quad \textit{Ans.} \tag{2}$$

Note that the number of degrees in the angle θ need not be deter-

mined, as its functions can be expressed in terms of the dimensions of the wall crane.

Note also that by taking the axis of rotation for equation (3) at A, two unknown forces were eliminated from the equation, simplifying the solution.

PROBLEMS

6–1. A steel bar AB, supported as in Fig. 90, is 40 in. long and weighs 9 lb. The angle with the horizontal is 35°. A weight of 15 lb hangs at point C, 12 in. from A, and a weight of 5 lb hangs at a point D, 24 in. from A. Set up the three equations for equilibrium and solve for the H and V reactions at A, and for the H reaction 6 in. from B.

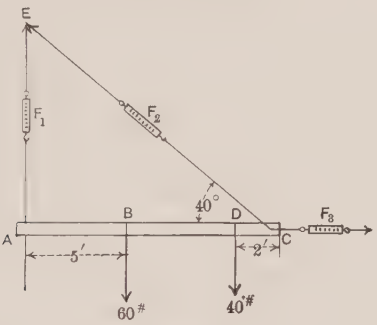


FIG. 97.

6–2. AC, Fig. 97, is a uniform steel beam, 12 ft long, weighing 20 lb. By use of equation (3),

(*a*) Take moments about A and thus find F_2.
(*b*) Take moments about C and thus find F_1.
(*c*) Take moments about E and thus find F_3.
(*d*) Resolve F_2 [found in (*a*)] into vertical and horizontal components and see if the *three equations*, $\Sigma H = 0$, $\Sigma V = 0$, and $\Sigma M = 0$, are satisfied by the numerical values now found.

6–3. Find the tension in BC and the hinge reactions for the wall crane, Fig. 96, when a load of 400 lb is applied (*a*) 1 ft from A, (*b*) 2 ft. from A, (*c*) 3 ft from A.

6–4. If the friction of the ladder against the building in Fig. 95 is 50 lb, find the horizontal reaction at A, and the amount and direction of this ground reaction at B.

6–5. In a davit, the dimensions, weight of the davit and load at C are as shown in Fig. 98. Compute the vertical force at A and the horizontal forces at A and B.

6–6. A locomotive reverse lever latch handle is shown in Fig. 99. The pin about which it rotates is at B. When a force of 30 lb is applied hori-

zontally at C, 1 in. from the end of the handle, what upward pull is exerted at A? What is the amount and direction of the force on the pin at B?

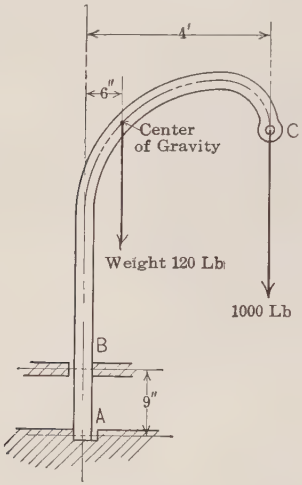


Fig. 98.

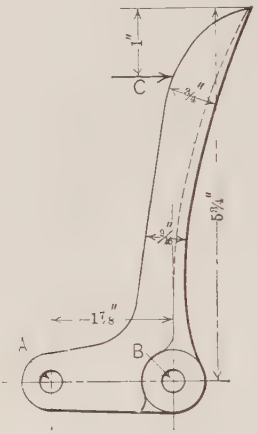


Fig. 99.

6–7. A door, 6 ft 6 in. × 3 ft 6 in., Fig. 100, is hung on hinges at B and C, 4 ft 6 in. apart; the door weighs 63 lb. Find the vertical and horizontal forces at each hinge. (Assume all the weight of the door to come on the upper hinge.)

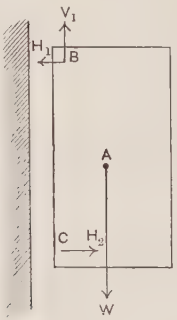


Fig. 100.

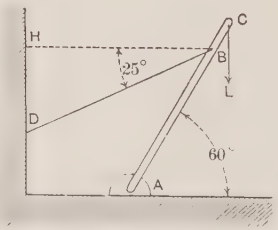


Fig. 101.

6–8. Find the tension in BD and the pin reactions at A for the pole and tie shown in Fig. 101, when load $L = 500$ lb, $AB = 5$ ft, $BC = 1$ ft. Assume the weight of the pole to be 80 lb, and its center of gravity to be $2\frac{1}{2}$ ft from A.

6–9. A bar AB, 6 ft long and weighing 50 lb, is suspended by a bracket on the wall at A and by a wire from B to a point C several feet higher on the wall. The angle at B between wire and bar is 30°. A weight of 500 lb is hung from

the bar at D 1 ft from B. Find the tension in the wire, the compression in the bar, the H and V reactions at A and the H and V reactions at C.

6–10. In a bracket similar to Fig. 38, AB is 28 in. long and weighs $2\frac{1}{2}$ lb, BC weighs $2\frac{1}{2}$ lb, and the angle at B is 45°. A load of 35 lb is supported on AB, resting equally on two points, one 7 in. from A and the other 7 in. from B. There is no load on the hook at B. Find the tension in AB, the compression in BC, the H and V reactions at A, and the H and V reactions at C.

SUMMARY

Non-concurrent forces are those whose lines of action are not all parallel and do not all pass through a common point.

The conditions for equilibrium are:

$$\Sigma H = 0 \tag{1}$$

$$\Sigma V = 0 \tag{2}$$

$$\Sigma M = 0 \tag{3}$$

In general, *for any combination*, equilibrium requires that there shall be no unbalanced force to produce change in translation, and no unbalanced moment to produce change in rotation. The conditions stated must, therefore, be true in *all* cases. *Concurrent forces* are, from this point of view, a special case in which, as the forces can have but a single resultant, if any, the first and second conditions for non-concurrent forces are sufficient, as the third reduces to the form $0 = 0$. Likewise *parallel forces* are a special case in which summation along one axis only is possible.

A **free body** is thought of as removed from contact with all other external bodies and acted on by forces only. The forces are caused by the bodies from which it is separated, and by its weight (the action of gravity).

The procedure in the solution of non-concurrent force systems ordinarily includes the following steps:

Select the member to be treated as a free body.

Draw or sketch a force diagram, showing all known and unknown forces, and their direction if known.

Set up the three equations, and solve for the unknown forces.

Note that the solution of the equations can sometimes be simplified by the choice of free body, and also by the choice of the center of rotation for the moment equation. Often this equation may be set up with only one unknown and solved first, thus making it unnecessary to solve by simultaneous equations for two or three unknowns.

CHAPTER VII

TRUSSES AND OTHER STRUCTURES

Many appliances and structures of industry and commerce are made up of parts or members riveted or pinned together at the joints. They are designed to support rigidly the loads applied to them. When in use, therefore, the loads acting at any point or points set up such tensions and compressions throughout the structure that the forces applied to each particular part or member are balanced and the member is thus held in equilibrium. **Any selected member may, therefore be considered as a "free body" as in Art. 43, and the values of the forces acting upon it may be determined by applying the three equations derived from the conditions of equilibrium for the system of forces acting.** The precautions to be observed in such solutions are to consider only the forces *acting upon* the member under consideration, *not* the forces exerted by it, and to make sure that each of these forces is given its proper direction.

Often the weights of the parts of the structure must be taken into consideration. This may be done most conveniently in many cases by regarding the weight as a single force acting at the center of gravity of each member. In other instances it is more convenient to resolve it into two forces regarded as acting one at each end of the member. Where the member is uniform and the center of gravity at the middle, the force at each end is one-half the weight of the member. Where the center of gravity is not at the middle point of the member, the part of the weight to be assumed as acting at each end must first be computed from the position of the center of gravity.

When considering the application of the laws of equilibrium to machines and structures, the student need not feel confused as to which of the several processes previously described he should use. Let him remember that there is *one method* which *can* be used for practically any case. **That method is the resolution of the forces into horizontal and vertical components and the application of the three equations: $\sum H = 0$, $\sum V = 0$, and $\sum M = 0$.** It is true

that this method is not always the simplest, or always the easiest. Experience will indicate when some other process may be substituted. With only three forces the force triangle may be preferable, particularly if it includes a right angle. Again the graphic method (parallelogram or polygon) may be selected. In some cases the moment equation may be used to distribute the forces to the joints and then the graphical solution followed.

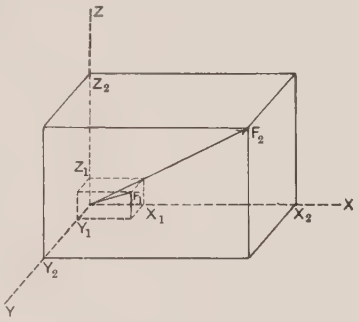


Fig. 102.

45. Forces not all in one plane. When the forces do not all lie in the same plane, the mathematical conditions for equilibrium of *concurrent* forces in one plane may be applied by the extension of the principle to include *three* rectangular axes, X, Y, and Z. See Fig. 102. Then the sum of the components along each axis must equal zero.

For a system of *non-concurrent* forces not in one plane the additional condition of the sum of all moments about any axis of rotation is required.

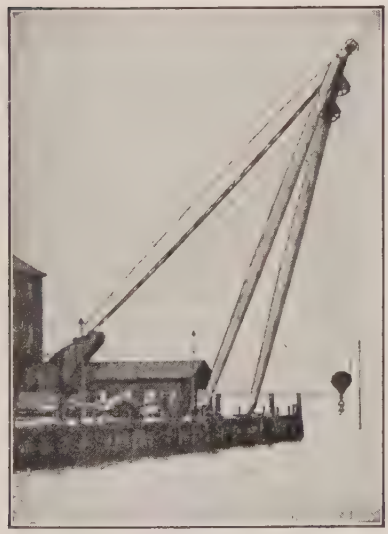


Fig. 103. Dock shears.

Most practical problems in statics are in two dimensions, or where not so originally, may readily be reduced to problems in two dimensions, see Art. 46; hence the method of resolution into components along three mutually perpendicular axes is seldom of particular value.

46. Shear legs. Figure 104 shows a model of shear legs for use in laboratory study. The feet are tapered and set in holes in the board screwed fast to the table top. Several quick adjustments of the spread of the legs, which are joined by a hinge at A, are thus possible. By adjusting the length of the tie AB,

the legs may be inclined at any desired angle with the vertical. The method of solution is as follows:

A known load is added at W. To this is to be added one-half the combined weight of the two legs AC and AD. If now we imagine the legs AC and AD replaced by a single imaginary leg from A to a point halfway between C and D, and therefore in the same vertical plane as the tie BA and the direction of the load or AW, we have a system of three forces meeting at a point at A. As the force in the direction AW is known, we may measure the necessary angles and determine, either by the graphical or the mathematical method, the tension in the tie and the compression in this imaginary single leg. Using the same method, the compression in the imaginary leg may be resolved into its components along the legs AC and AD.

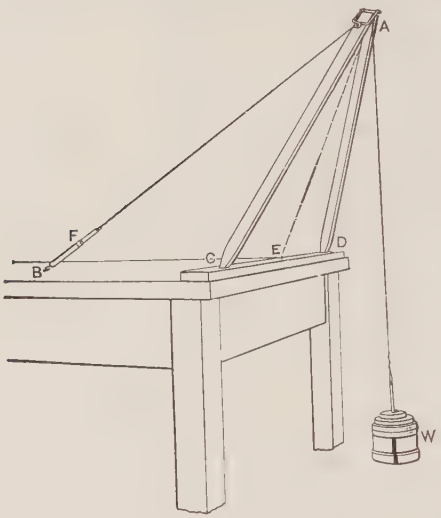


Fig. 104. Model shear legs.

Example.

Find the force in the tie and in each leg of the shears in the diagram of Fig. 104 given the following data: load $W = 184$ lb; weight of each leg $= 10$ lb; angle $WAB = 67°$; $WAE = 25°$; $CAE = DAE = 31°$.

Solution. Consider the three forces at A: $W = 184$ lb + one-half the weight of each leg, the tension T in the tie AB, and the compression C in an "imaginary" leg AE in the plane of the load and the tie, and also in the plane of AC and AD. Then the two equations will be:

$$\Sigma H: \quad C \sin 25° - T \sin 67° = 0 \tag{1}$$

$$\Sigma V: \quad C \cos 25° - T \cos 67° - (184 + 5 + 5) = 0 \tag{2}$$

Looking up the trigonometrical functions and substituting:

$$0.423\ C - 0.921\ T = 0 \tag{1}$$

$$0.906\ C - 0.391\ T - 194 = 0 \tag{2}$$

From equation (1) express the value of C in terms of T and substitute in equation (2):

$$0.423\ C = 0.921\ T$$

$$C = \left(\frac{0.921}{0.423}\right) T = 2.177\ T$$

$$0.906 \times 2.177\ T - 0.391\ T - 194 = 0$$

$$(1.972 - 0.391)\ T = 194$$

$T = 122.6$ lb in the tie AB *Ans.*

Substitute in equation (1) (transposed):

$$C = 2.177\ T = (2.177)\ (122.6)$$

$C = 266.9$ lb in the imaginary leg AE

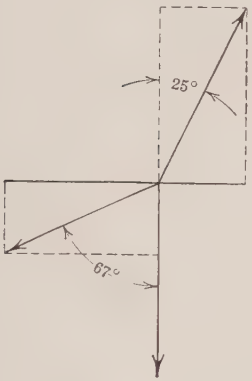


FIG. 105.

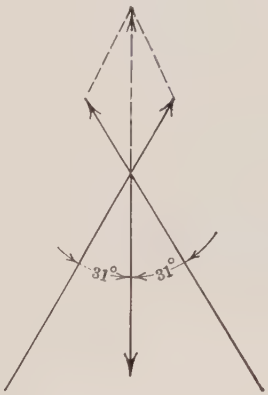


FIG. 106.

Now consider the three concurrent forces at A in the plane of the legs. In that plane force C in the imaginary leg is vertical and is held in equilibrium by the thrust of AC and of AD. Call this thrust C_1 for each, since they are symmetrical and equal.

The equation for the vertical components only is necessary.

$$\Sigma V: \quad +C_1 \cos 31° + C_1 \cos 31° - C = 0$$

$$0.857\ C_1 + 0.857\ C_1 = 266.9$$

$$C_1 = \frac{266.9}{2 \times 0.857}$$

$C_1 = 155$ lb in each leg *Ans.*

If desired, this thrust can be resolved into its horizontal and vertical reactions at the ground.

In practice the load rope continues over a pulley at A causing an additional force at A; therefore an additional term (of known amount) is required in each of the equations.

47. Water-tank supports. The common water tank shown in Fig. 107 furnishes another illustration of equilibrium with concurrent forces having components in three axes X, Y, and Z. A simple solution because of the ease with which dimensions, angles, etc., may be obtained is suggested in Fig. 108. The weight W of the tank and contents is supported by the reactions of the four legs. If continued, these five forces will intersect at a common point at O. If now we assume forces OA and OB replaced by a single force OM, and OC and OD by the single force ON, we have a system of concurrent forces in a plane. The required forces OM and ON may now be obtained graphically or algebraically in the usual way, and OM and ON may then be resolved into their components along OA and OB, and OC and OD, respectively. These are the compressions in the supporting legs.

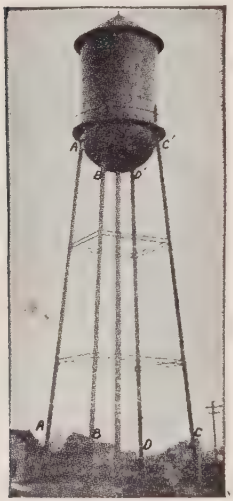


FIG. 107. Tank of the city water works, Sycamore, Ill.

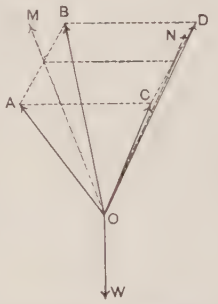


FIG. 108. Force diagram for legs of a water tank.

48. Cranes and derricks. Figure 109 shows a model of a derrick with guy-ropes, suitable for laboratory tests. Spring balances are inserted in the tie and guy-ropes to check the computed tensions in these members, and balances are also arranged to check the reactions at the foot of the boom. The parts are purposely made heavy so that the effect of their weight on the distribution of forces in the members may be studied, and provision is made for changing the angles of the apparatus, and the points of application of the tie and load as desired.

In the computation of stresses the load F and the weight W of the boom, acting at its center of gravity G, are assumed as the

only known forces, and the angles and distances are measured as required. From the conditions for equilibrium for the system of

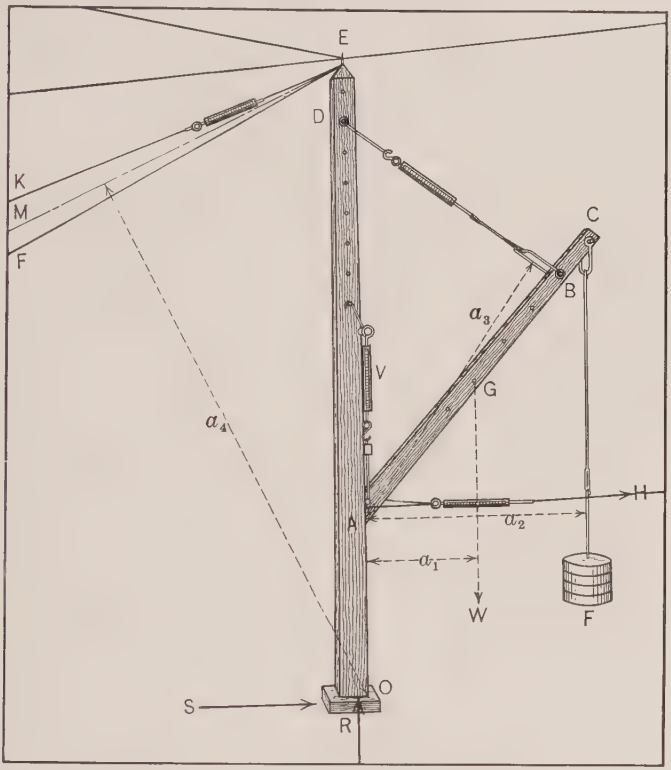


FIG. 109. Laboratory model of a derrick.

forces acting, taking *the boom AC as the free body* and speaking of the tension in the tie BD as the force T, the three equations are:

$$\Sigma H: \quad H - T \sin \angle BDA = 0 \tag{1}$$

$$\Sigma V: \quad V + T \cos \angle BDA - W - F = 0 \tag{2}$$

$$\Sigma M: \quad W \times a_1 + F \times a_2 - BD \times a_3 = 0 \tag{3}$$

These equations give the tension in BD and the reactions at A. The computed values may be checked by the balance readings. Balances V and H should be read when the foot A of the boom is held just free from the mast by a horizontal pull.

At E the forces are not in one plane. Imagine the guy-ropes

EK and *EF* replaced by a single guy *EM* in the plane of *EK* and *EF* and *also* in the vertical plane with the axis of the mast and the load *F*. This single tie will support the mast, provided the boom does not swing out of the plane of the tie and mast. Therefore, taking moments about *O* for the *whole derrick as a free body,* when there is no pull *H* at *A*, we have the equation

$$\Sigma M: \quad W \times a_1 + F \times a_2 - EM \times a_4 = 0$$

from which the tension in the imaginary guy *EM* may be found. This tension may then be resolved into its components along *EF* and *EK*.

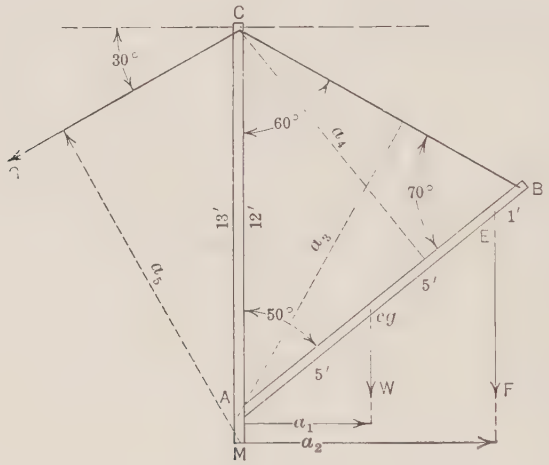


Fig. 110.

The reaction *R* at the foot of the mast is obviously equal to the weight of the whole crane + *F* + vertical component of *EM*. The reaction *S* (when there is no horizontal pull applied at *A*) may be found from the equation of moments about *E*,

$$W \times a_1 + F \times a_2 - S \times OE = 0$$

It is also equal to the horizontal component of the tension *EM*. Why?

The compression in the various sections of the mast may be computed from the vertical components of the forces *above* the sections plus the weights.

Example.

Given a derrick, with dimensions as shown in Fig. 110, the weight

of mast 200 lb, the weight of boom = W = 100 lb, and the load F = 2000 lb, to find

(*a*) the force T in tie BC
(*b*) the H and V reactions of boom at A
(*c*) the force C in boom AB
(*d*) the force in guy-wire G
(*e*) the H and V reactions at foot of mast M

Solution. Using the boom AB as the *free body* set up the three equations as follows:

$$\Sigma H: \quad -T \cos 30^\circ + H_a = 0 \tag{1}$$

$$\Sigma V: \quad +T \sin 30^\circ + V_a - 2000 - 100 = 0 \tag{2}$$

ΣM (about A):

$$+100 \times a_1 + 2000 \times a_2 - T \times a_3 = 0 \tag{3}$$

Solve for the three moment arms:

$$a_1 = 5 \cos 40^\circ = 5 \times 0.766 = 3.83$$
$$a_2 = 10 \cos 40^\circ = 10 \times 0.766 = 7.66$$
$$a_3 = 11 \sin 70^\circ = 11 \times 0.940 = 10.34$$

Substitute in equation (3) and solve:

$$+100 \times 3.83 + 2000 \times 7.66 - T \times 10.34 = 0$$

$$T = \frac{383 + 15{,}320}{10.34} = \frac{15{,}703}{10.34} = 1520 \text{ lb in tie } BC \quad \textit{Ans.} (a)$$

Substitute in equation (1) and solve:

$$-1520 \times 0.866 + H_a = 0$$
$$H_a = 1520 \times 0.866 = 1315 \text{ lb, horizontal at } A \quad \textit{Ans.} (b)$$

Substitute in equation (2) and solve:

$$+1520 \times 0.500 + V_a - 2100 = 0$$
$$V_a = 2100 - 760 = 1340 \text{ lb, vertical at } A \quad \textit{Ans.} (b)$$

Now, *still using the boom as the free body,* set up a moment equation about C as a center:

$$\Sigma M: \quad +100 \times a_1 + 2000 \times a_2 - C \times a_4 = 0$$

The arm: $a_4 = 12 \sin 50^\circ = 12 \times 0.766 = 9.19$

Then: $+100 \times 3.83 + 2000 \times 7.66 - C \times 9.19 = 0$

$$C = \frac{383 + 15{,}320}{9.19} = \frac{15{,}703}{9.19} = 1710 \text{ lb in } BC \quad \textit{Ans.} (c)$$

Now take *the mast, tie, and boom together as the free body* and set up the three equations, using center of moments at M:

$$\Sigma H: \quad -H_g + H_m = 0 \tag{1}$$

$$\Sigma V: \quad -V_g - 100 - 200 - 2000 + V_m = 0 \tag{2}$$

$$\Sigma M: \quad +100 \times a_1 + 2000 \times a_2 - G \times a_5 = 0 \tag{3}$$

The arm: $a_5 = 13 \sin 60° = 13 \times 0.866 = 11.25$

Substitute in equation (3) and solve:

$$+100 \times 3.83 + 2000 \times 7.66 - G \times 11.25 = 0$$

$$G = \frac{383 + 15{,}320}{11.25} = \frac{15{,}703}{11.25} = 1400 \text{ lb in the guy-wire} \quad \textit{Ans.} \ (d)$$

Resolve G into its components and substitute in equations (1) and (2).

$$H_g = G \cos 30° = 1400 \times 0.866 = 1210$$

$$V_g = G \sin 30° = 1400 \times 0.500 = 700$$

$$-1210 + H_m = 0 \tag{1}$$

$$H_m = 1210 \text{ lb horizontal at } M \quad \textit{Ans.} \ (e)$$

$$-700 - 100 - 200 - 2000 + V_m = 0 \tag{2}$$

$$V_m = 3000 \text{ lb vertical at } M \quad \textit{Ans.} \ (e)$$

H_m could also be determined by use of the moment equation about C, thus:

$$\Sigma M: \quad +100 \times 3.83 + 2000 \times 7.66 + 200 \times 0 - H_m \times 13 = 0$$

$$H_m = \frac{15{,}703}{13} = 1210, \text{ as before}$$

Several optional methods suggest themselves which may be used for a check, or as substitutes. One is to employ the *point* system, by distributing $\frac{10}{11}$ of the 2000 lb, and $\frac{5}{11}$ of the 100 lb at B and the remainder at A; then, solving the two concurrent systems, at B and at A, either graphically or mathematically.

The problem as given lacks one feature that is almost always encountered in practice, that of the continuation of the hoisting rope holding the 2000-lb load down along the boom to A, or to a point on the mast somewhat above A, as D (not on diagram). If the load is suspended by a single pulley with two ropes, each rope carries 1000 lb, and the rope continuing along the boom will increase the force in the boom 1000 lb. This will affect the force in the tie and the reactions at A, which can be found by setting

up the three equations as before but with the additional term in each. Such an extension of this problem has been computed with the 1000-lb force in the rope going to a point D, 1 ft above A. The results were: force in tie, $T = 1450$ lb; horizontal at $A = 2060$; vertical at $A = 1960$.

PROBLEMS

7–1. In a shears similar to Fig. 104, if the load equals 300 lb, the weight of each leg 20 lb, angle of tie with the vertical equals 70°, angle of the imaginary leg with the vertical equals 20°, and the angle between the legs equals 60°, find the force in the tie and that in each leg. Solve mathematically and check graphically.

7–2. The legs of the shears, Fig. 103, are each 80 ft in length, with a spread of 20 ft at the feet and a vertical height from the dock to the tip of 60 ft. The tie and hoisting chain both run from the top of the shears to a point 40 ft back of the middle point of the line between the feet. Find the stresses in the members when a load of 20 tons is being raised, if the tension in the hoisting chain winding on the drum is one-third of the load.

7–3. In a derrick similar to Fig. 110 the mast CM is 26 ft high, the boom AB is 22 ft long, A is 2 ft above M, center of gravity is 10 ft from A, E is 2 ft from B, angle BAC equals 50°, each of angles ACB and ACG equals 60°, the weight of mast equals 200 lb, weight of boom equals 100 lb, the load equals 1000 lb. Find the forces in tie BC and boom AB and guy CG; also the H and V reactions at A and at M.

7–4. In the derrick of Fig. 110, if the load F is 1000 lb, the weight of mast 200 lb, the weight of boom 100 lb, and all other data the same as in the diagram, find the forces in tie and boom and guy, and the reactions at foot of boom and at foot of mast.

7–5. Referring to the derrick of Fig. 110, if the load of 2000 lb is carried on two ropes, one of which passes over a pulley at E and continues to a point D, 1 ft above A, find the tension in the tie BC and the H and V reactions at A.

7–6. The mast CA of the derrick, Fig. 60, is $14\frac{1}{2}$ ft long, the boom CD is 20 ft long. The guys AF and AG are each 18 ft long, with a spread of 8 ft at the feet. The distance AE is 4 in., and the boom CD weighs 320 lb. The hoisting rope from D to B, not visible in the figure, runs parallel to the boom. Find the tension in the tie DE and in the guys AF and AG, also the thrust of the boom at C, when the load L is 2 tons and the boom makes an angle of 45° with the mast and its vertical plane bisects the angle GAF.

49. Bridge and roof trusses. Figures 111, 112, and 113 show typical forms of simple trusses as used in bridge construction, etc. The stresses in the members of trusses of this sort may be found by taking *each joint in turn as a free body* and applying to each the equations for equilibrium for a system of concurrent forces. Thus,

suppose the Pratt truss, Fig. 113, bears loads which are expressed in *tons* by the figures placed near the arrows; required to find the stress in each member of the truss and to determine in each case whether the member is under tension or compression.

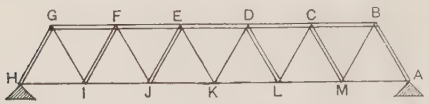


FIG. 111. The Warren Truss.

FIG. 112. The Howe Truss.

(*a*) **Pier reactions:** The sum of the loads is 230 tons, and since the truss is symmetrically loaded, the reactions at the piers must each be 115 tons.

(*b*) **Forces at the point *A*:** At *A* *three forces* act, the upward reaction *R*, now known to be 115 tons, a force from the member *BA*, which from inspection we should expect to be a thrust toward *A*, and a force from the member *AH*, which we will assume is

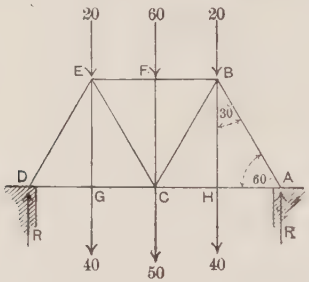


FIG. 113. The Pratt Truss.

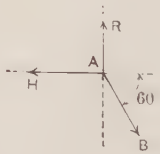


FIG. 114.

under tension, and therefore pulling *A* toward *H*. Considering the joint at *A* as a free body, and lettering the forces in the same way as the members, we have the force diagram shown in Fig. 114. Therefore the sum *H* and sum *V* equations are:

$$\Sigma H: \quad BA \cos 60^\circ - AH = 0$$

$$\Sigma V: \quad 115 - BA \sin 60^\circ = 0$$

Solving $BA = 132.8$ tons

and $AH = 66.4$ tons

These equations check in every way and we may note that our first assumptions were correct, and that the member *AB* is push-

ing against A (and therefore also against B), and hence is under *compression*; the member AH is pulling on A (also on H) and is therefore under *tension.*

(*c*) **Forces at the point *H*:** Next choose the point H and draw its force diagram (Fig. 115). Remember that if a member of any structure is under tension, *it must pull equally and in opposite directions upon the two joints* of the structure between which it is attached, or if under compression, *must push equally upon the two joints.* Every time we determine the stress in a member from a consideration of the conditions at one end, we also determine the action which it exerts upon the joint at the other end. Inspection shows that for equilibrium HC must be equal and opposite to the known tension $HA = 66.4$ tons, and also that the member HB must be pulling up from H with a force equal to the 40-ton load at H.

$$\therefore \quad CH = 66.4 \text{ tons (tension)}$$
$$HB = 40 \text{ tons (tension)}$$

Fig. 115.

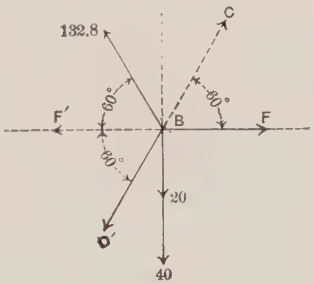


Fig. 116.

(*d*) **Forces at the point *B*:** Next take the point B. Its force diagram may be drawn as in Fig. 116. It is known from the previous solutions that the member AB pushes *up* and to the *left* with a force of 132.8 tons.

There is also the downward load of 20 tons and the tension of the member $BH = 40$ tons (see point H). There are two other forces acting at B whose amounts are as yet unknown; one from member CB, along the line we will call CC' in the force diagram, and the other from the member FB along the line $F'F$. We will call these forces BC and BF, but assume that as yet we do not know which way they act; i.e., whether members CB and

FB are compression or tension members. Assume the force BC to be upward and to the right (i.e., compression) and the force BF to be to the right (compression), and write the equations for the point B

$$\Sigma H: \quad -132.8 \cos 60 + BC \cos 60 + BF = 0$$

and

$$\Sigma V: \quad -40 - 20 + 132.8 \sin 60 + BC \sin 60 = 0$$

Solving the ΣV equation:

$$BC = -63.5 \text{ tons}$$

The negative sign shows that the force along BC must be such as to give a negative value for its V component, i.e., must be in direction BC'; or, in other words, BC must be a tension member instead of a compression member as assumed.

Correcting the sign for BC (i.e., the direction) in our ΣH equation, and solving for FB, we obtain

$$FB = +98.2 \text{ tons}$$

Here the positive sign shows that the member FB acts to the right at B, as we assumed in writing the equations, and BF is therefore a compression member.

(*e*) **Forces at the point *F*.** Next take the point F. From symmetry we know

$$EF = BF = 98.2 \text{ tons compression}$$

The only unknown is the member CF which inspection shows must be in compression and equal to 60 tons to balance the load at F. Since the other half of the truss is loaded in the same way, the forces in all the members are now known, but as a check we may write the equations for the point C.

The vertical forces are given by the equation:

$$\Sigma V: \quad -50 - FC + BC \cos 30 + EC \cos 30 = 0$$

In which: $$BC = EC = 63.5$$

Then: $$-50 - 60 + 2 \times 63.5 \times .866 = 0$$

This reduces to $-50 - 60 + 110 = 0$, thus checking all the previous calculations.

Figures 117 and 118 show bridge trusses in actual use, the stresses for which, under given conditions of load, may be determined as explained for the truss of Fig. 113.

FIG. 117. Through span over the Wabash-Pittsburgh terminal railway.

FIG. 118. Baltimore and Ohio Railroad bridge crossing south branch of Potomac River in West Virginia.

Such structures are subject to both a " dead " load, from the weight of the parts, ties, track, etc., and a varying or " live " load, coming from the weight of the engine, cars, loaded wagons, etc., hauled over them. These loads may be applied at the joints by the method of construction, or, if not, the distributed load should be considered concentrated at the joints for the purposes of computation, just as in previous instances we have allowed for the weight of a part by assuming a half-weight as acting at each end.

Most types of trusses can be solved by the *point* method. It should be noted, however, that with only the two equations, $\Sigma H = 0$ and $\Sigma V = 0$, only two *unknown* forces at a joint are determinable. For this reason the reactions at the end supports must usually be determined first and the joint at a support taken as the first point. In Fig. 119, for example, the joint at A or G must be chosen as the first point. From this first point, we may proceed from joint to joint in any order so long as the condition of not more than two unknown forces at a point is met.

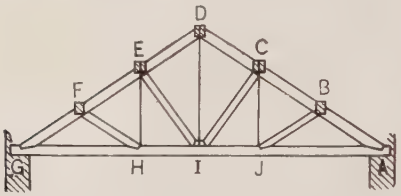


Fig. 119. Timber roof truss.

The reactions of the supports may be determined from inspection in the case of symmetrical trusses which are symmetrically loaded. In other instances, these reactions must be computed by taking moments about the supports, as for parallel forces.

For **roof trusses** the " dead " loads consist of the weight of the truss and the weight of the section of the roof supported. The " live " loads are the wind and snow loads. These may be applied to the joints by an auxiliary piece in the construction of the truss or in any case may be carried to the joints or apex for the purposes of computation as follows: Let the shaded area $MNOP$ in Fig. 120, which is a top view of a section of the roof, represent the portion of the roof and snow load taken by the truss AEC. This area will extend halfway to the trusses on either side, and the total weight of roof and snow will be equal to the weight per square foot times area $MNOP$. Of this, the weight of the half area $MQRP$ comes on the member AE, and the weight of the other half area $QNOR$ comes on EC. We may therefore regard the load on AE as the total weight W of the area $MQRP$ *acting at the center of gravity of the area*, which is the *middle point of* AE, or what is the same thing, as acting $\frac{1}{2}$ at E and $\frac{1}{2}$ at A. The load on AE is therefore equivalent to $\frac{1}{2}\ W$ at E and $\frac{1}{2}\ W$ at A. In the same way the load on EC

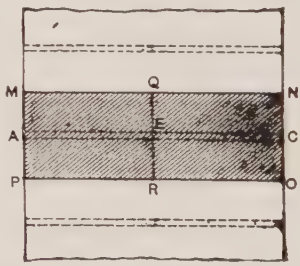


Fig. 120. Top view of section of roof. Load on the shaded area is carried by AEC.

is $\frac{1}{2}\ W$ at E and $\frac{1}{2}\ W$ at C. The total apex load at E is therefore $\frac{1}{2}\ W + \frac{1}{2}\ W = W$, and the loads at A and C are each $\frac{1}{2}\ W$. The student should note that the loads at A and C produce pressure on the supports but *have no effect in producing stress in the members AE and EC* of the truss.

The solution of a truss can also be accomplished by the **method of sections.** This consists in passing a section at any convenient point to separate a member, or a joint, or a group of members into a **free body.** The part of the truss which is considered as the free body is acted upon by the external forces on that part and by the *stresses* in the members cut by the section.

One advantage of this method is that the stress in any particular member can be determined directly, without beginning at one end as is necessary in the method of joints, and computing all the members until the desired one is reached.

Example.

Take the Pratt truss of Fig. 113. Let a section be passed cutting members *EF*, *EC*, and *GC*. The part of the truss on either side of the section is a free body in equilibrium. Consider the left-hand part as shown in Fig. 121, which is in equilibrium under the action of the 20- and 40-ton loads, of the 115-ton reaction, and of the stresses in members *EF*, *EC*, and *GC*. These stresses can be computed by the use of the *three equations.*

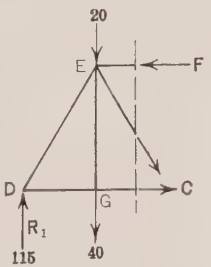


FIG. 121. Section of truss of Fig. 113 as a free body.

Solution. A little consideration shows that the ΣV equation will be simple to use to determine *EC*, since the other members have no vertical components. The ΣM equation can well be used for the other members, taking the center of rotation first at *C*, giving two unknowns a zero moment arm and then at *E*, for the same reason.

The equations are as follows:

$$\Sigma V: \quad -EC \sin 60° - 20 - 40 + 115 = 0$$

$$EC = \frac{55}{0.866} = 63.5 \text{ tons} \quad Ans.$$

Since the dimensions are not given, let h represent the vertical height; then the horizontal distance between joints will be $h/\tan 60°$.

$$\Sigma M \text{ (at } C): \ -(EF \times h) - \left(60 \times \frac{h}{\tan 60°}\right) + \left(115 \times \frac{2\,h}{\tan 60°}\right) = 0$$

$$-EF \times h \tan 60° - 60\,h + 230\,h = 0$$

$$EF = \frac{170}{\tan 60°} = \frac{170}{1.73} = 98.2 \text{ tons} \quad Ans.$$

ΣM (at E): $$-GC \times h + 115 \times \frac{h}{\tan 60°} = 0$$

$$GC = \frac{115}{1.73} = 66.4 \text{ tons} \quad Ans.$$

50. Graphical methods. The graphical solution offers a convenient method for checking truss computations, as well as making a rapid approximate determination.

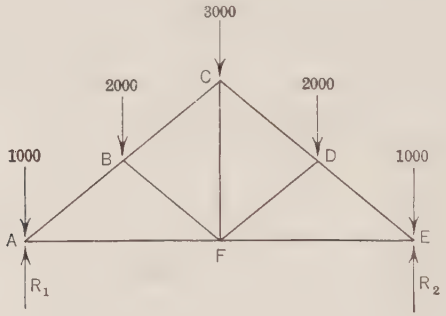


Fig. 122.

Example.

Consider the truss in Fig. 122, in which the span AE is 24 ft, the rise CF 10 ft, and joints B and D are at the middle of AC and CE, respectively. To determine the stress in each member graphically.

Solution 1. The reactions are equal, since the loading is symmetrical.

$$R_1 = R_2 = \tfrac{1}{2}(1000 + 2000 + 3000 + 2000 + 1000) = 4500 \text{ lb}$$

The force diagrams for joints A, B, and C are shown in Fig. 123, not to scale, all forces shown as pulls. If there is doubt about the direction shown for certain forces, the assumption can be verified by the corresponding force polygon in Fig. 124.

At point A, Fig. 124, the reaction is first drawn to scale. Let 1 in. = 2000 lb; then the vector will be $2\frac{1}{4}$ in. long, up, to represent 4500 lb. From its forward end, the 1000-lb load is drawn vertically down, followed by a line parallel to AB long enough to intersect a horizontal line parallel to AF, which will close the polygon. Place arrow heads pointing the same way around. Scaling the two unknown vectors, the force in AB is found to be 5500 lb, and that in AF 4200 lb. The directions show compression in AB and tension in AF.

At B draw a vector equal to 5500 lb for BA equal and opposite to its direction at point A. Then draw 2000 lb vertically down, then a vector parallel to BC intersecting one parallel to FB to close the polygon. If the arrow heads are placed the same way around, the directions are as shown in Fig. 124(b). The load at B is supported by compression in AB and FB, and FB is supported by tension in FC, which member is supported from C. Scaling the unknown vectors gives forces of 3900 lb and 1600 lb for BC and BF, respectively.

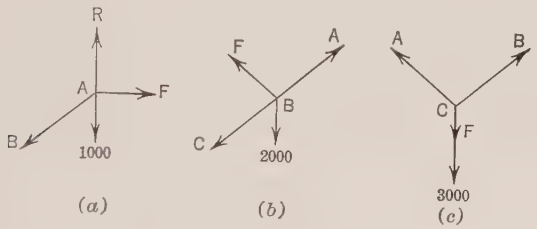


Fig. 123.

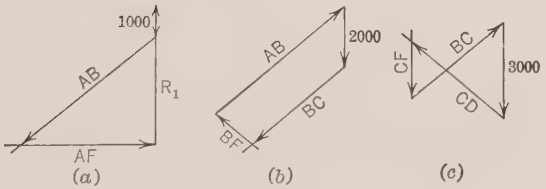


Fig. 124.

At C draw 3900 lb opposite to its direction at B, then the 3000-lb load, and complete the polygon with vectors parallel to CD and CF, as in Fig. 124(c), scaling 3900 lb and 2000 lb, respectively. CF, in tension, acts downward from point C, as determined by closing the polygon.

The truss loading being symmetrical, no further joints need be considered.

The foregoing solution can be simplified and somewhat shortened by the following **graphical analysis,** which combines the three polygons of Fig. 124 into one diagram and saves duplication of vectors by utilizing each line twice. For this method another system of lettering the truss is used, as shown in Fig. 125. Here the *spaces* instead of the joints are lettered. A force is named by the letters of the spaces on either side of it. Thus the 1000-lb load at the left support is called AB, the reaction is GA, the

stresses in the two members are BH and HG, respectively. Observe that the spaces are lettered in clockwise order around the outside of the truss. The joints are sometimes designated by number in this system.

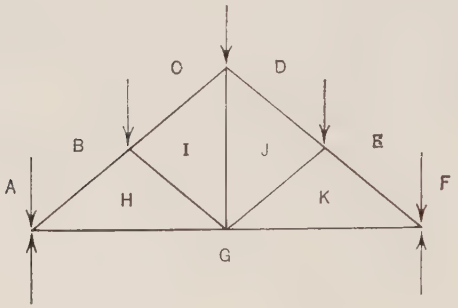


Fig. 125.

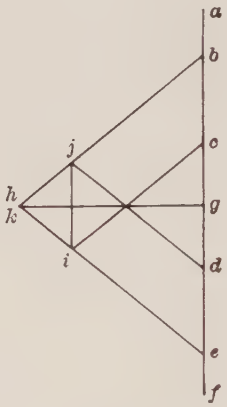


Fig. 126.

Solution 2. The stress diagram is now constructed to scale, using small or lower-case letters corresponding to the capitals of the space diagram. First draw the load line, as in Fig. 126, by plotting load AB as ab (let 1 in. = 2000 lb, then $ab = \frac{1}{2}$ in.), BC as bc, etc., including the two reactions fg and ga up to the point of beginning.

Next plot the forces at the left reaction, joint 1. Taking them in clockwise order, ga is already drawn, so is ab; draw bh parallel to BH and hg parallel to HG. Observe that this polygon $gabhg$ corresponds to the polygon in Fig. 124(a). Observe also that, reading clockwise around the *space* diagram the forces are GA, AB, BH, and HG, and that the *direction* of each force applied to this joint is given by naming the vector with its letters *in the same order*, that is, ga, ab, bh, and hg. Thus hg (not gh) acts to the right from joint 1, exerts a pull on the joint, and is therefore in tension.

Take joint 2 next. The forces are HB, BC, CI, and IH. The polygon is $hbcih$, in which hb and bc are already drawn.

Take joint 3. The forces are IC, CD, DJ, and JI. The polygon is $icdji$.

It is unnecessary to analyze the truss further, but if desired the stress diagram may be completed by the polygons $jdekj$, $kefgk$, and, as a final check, $kghijk$.

Suppose that it is desired to determine whether some particular member as IJ is in tension or compression. Considering the forces at joint 3 and naming them by taking the letters in clockwise order, the force in question is JI. On the stress diagram ji reads downward; therefore the force is down *from* the joint and is a pull or tension. This can be checked by considering joint 6. In clockwise order the force here is IJ and on the stress diagram ij reads upward; therefore the force is a pull up *from* this joint and is in tension, as previously determined.

In the foregoing example the reactions were determined most simply by computation. If it is desired to determine the reactions graphically, the method of the **equilibrium polygon** may be used.

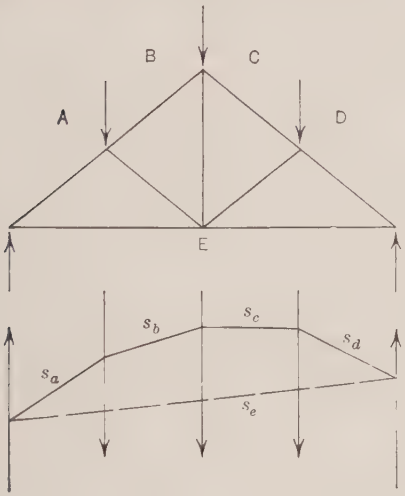


Fig. 127(*a*).

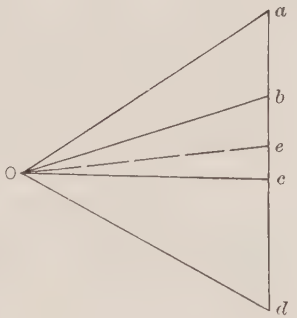


Fig. 127(*b*).

Example.

Consider the truss of 24-ft span, in Fig. 127. The loading is 2 tons at AB, 2 at BC, and 3 at CD. As it is unsymmetrical the reactions are not easily found by inspection. They may be computed by the moment equation. Thus

$$R_1 = \frac{2 \times 18 + 2 \times 12 + 3 \times 6}{24} = 3.25 \text{ tons}$$

$$R_2 = \frac{2 \times 6 + 2 \times 12 + 3 \times 18}{24} = 3.75 \text{ tons}$$

Solution. To check these results graphically proceed as follows.

(1) Draw the load polygon *abcd* as in Fig. 127(*b*). The *sum* of the two reactions will equal *da*.

(2) Take any convenient point *o*, not on the load line, as a *pole*. Draw *rays* from *o* to each point *a*, *b*, *c*, and *d*. This may be thought of as resolving each force into two imaginary components by which it could be replaced. For example, *ab*, *bo*, *oa* form the sides of a closed triangle in which *ab* could represent the resultant of *ao* and *ob*.

(3) Take a convenient point on the force line of *AB* (produced if desired) as in Fig. 127(*a*). From this point draw *strings* S_a and S_b, parallel to the rays *ao* and *ob*. S_a intersects the force line of *EA* (or R_1) and S_b intersects the force line of *BC*.

(4) From the point of intersection on the force line of *BC* draw the string S_c parallel to ray *oc* intersecting the force line *CD*.

(5) Draw the string S_d parallel to ray *od* intersecting the force line *DE*.

(6) The four strings now represent the corresponding components of the forces in the load or force polygon. They are parallel to the rays and proportional in length. If a string S_e be drawn *to close the equilibrium polygon*, then it will show the direction of one of the components of each reaction.

(7) Draw the ray *oe* parallel to the string S_e, thus locating the point *e* on the force diagram. The components of the reactions are *do* and *oe*, *eo* and *oa*. The reactions are *de* and *ea*.

(8) Scale *de* equals 3.75 and *ea* equals 3.25, from which the reactions *DE* and *EA* are determined, agreeing with the computed results.

For other applications, and more extensive consideration, of this principle, reference may be made to a more advanced text on mechanics or graphic statics.

PROBLEMS

7–7. In a simple Warren truss, Fig. 128(*a*), with each member 6 ft long

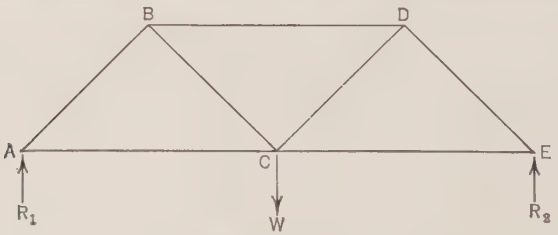


Fig. 128(*a*).

and a 1000-lb load at the joint at the center of the bottom chord, compute the reaction at each support and the stress in each member.

7–8. In a simple Warren truss, Fig. 128(*a*), the two lower chord members and the upper chord are each 6 ft long, each diagonal member is 4 ft long, the dead weight is 10 lb per linear foot for each member, the live load is 3000 lb at the joint at the center of the bottom chord. Compute the reactions and the stress in each member.

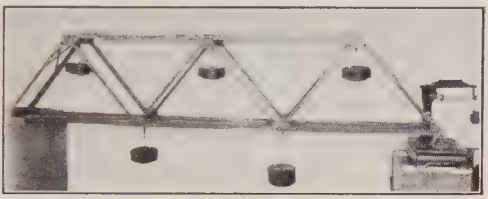


FIG. 128(*b*). Laboratory model of a Warren Truss.

7–9. Compute the end reactions and the stress in each member of the roof truss of Fig. 122, with loads as given on the diagram.

7–10. Compute the end reactions and the stress in each member of the roof truss of Fig. 122, if the loads are as follows: 1000 lb at joints *B*, *C*, *D*; 500 lb at joints *A* and *E*.

7–11. Find the stresses in the members of the Howe truss, Fig. 112, when the loads are 20 tons at each joint at the bottom and 5 tons at each joint at the top. Angle *BAL* is 60°, angle *ABL* is 30°.

7–12. Find the stresses in the members of the Pratt truss in Fig. 117, when a truck weighing 10 tons stands on the center of the bridge, and the dead load at each joint is $\frac{1}{2}$ ton. The angles of the truss are as in Fig. 113.

7–13. The trussed beam of Fig. 129 bears loads of 1500 lb each at *A*, *B*, and *C*. $BA = BC = 10$ ft. $BD = 2$ ft. Compute the stress in each member.

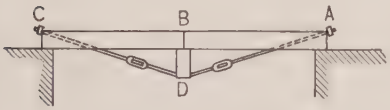


FIG. 129. Truss to stiffen a floor beam.

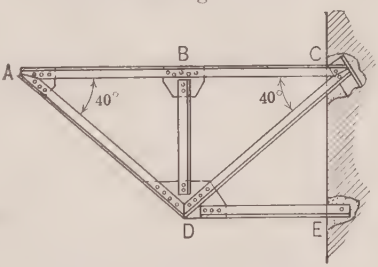


FIG. 130. Truss to support a balcony.

7–14. The roof truss of Fig. 119 bears loads of $\frac{1}{2}$ ton at each of the joints *A*, *B*, *C*, *D*, *E*, *F*, and *G*. The angle *BAJ* = the angle *BJA* = 30°. Compute the stress in each member due to these loads.

7–15. The balcony truss of Fig. 130 bears a load of 500 lb per running foot for *AC*. $AB = BC = 10$ ft. Compute the stresses in the members.

7–16. Find the stress in member *CD* of the Warren truss, Fig. 128(*a*), using the method of sections (or free body), if there is a load of 1000 lb on each of the joints *B*, *C*, and *D*.

7–17. Solve the Warren truss of Problem 7–7 graphically.

7–18. Solve the roof truss of Problem 7–10 graphically.

7–19. Solve the Pratt truss of Fig. 113 graphically.

7–20. A Warren truss of three panels instead of two as Fig. 128(*b*), is unsymmetrically loaded. The joints are lettered similarly to Fig. 128(*a*), from *A* at the left reaction to *G* at the right. If there is 1000 lb at *B*, 2000 lb at *C*, 2000 lb at *D*, 4000 lb at *E*, and 3000 lb at *F*, and each member is 6 ft long, find the reactions at *A* and *G* and the stress in each member.

7–21. Find the tension in the chains *AB* and *AD* and the force *R* for the stone tongs of Fig. 131, when lifting a block weighing 1500 lb, if $d_1 = 18$ in., $d_2 = 15$ in., angle $BAD = 140°$, distance $EC = 16$ in., and the pin *O* is the vertical line of *P*.

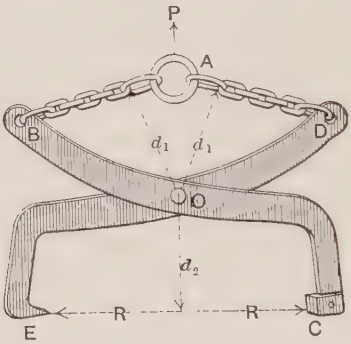


Fig. 131. Stone tongs.

7–22. The arch of Fig. 132 measures 10 ft horizontally from center to center of the foot blocks. The load is 2000 lb per foot of this horizontal line.

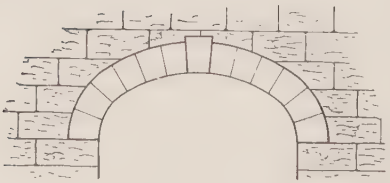


Fig. 132. Stone arch.

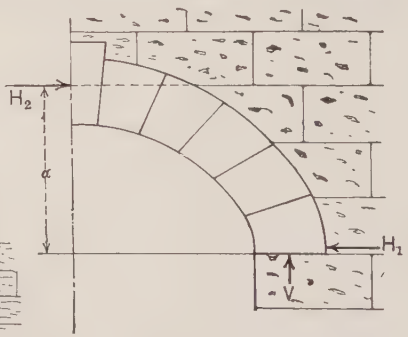


Fig. 133. Force diagram for a half arch.

The height of the arch axis at the middle, above this horizontal, is 2 ft. Treat one half the arch as a free body and compute the vertical and horizontal reactions at each foot block, and the horizontal thrust at the middle of the arch.

7–23. Find the force F necessary to push the 200-lb cart of Fig. 134 up

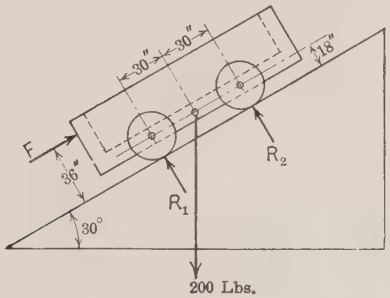


FIG. 134.

the slope, and the two forces R_1 and R_2 with which the slope pushes against the wheels of the cart, dimensions being as shown. Do not consider friction.

SUMMARY

The laws of equilibrium provide a method of solution for the stresses in members of machines, trusses, and other structures. These laws are expressed by the **three equations**, $\Sigma H = 0$, $\Sigma V = 0$, $\Sigma M = 0$. Alternative methods may be substituted for the solution by these equations in some special cases. Forces which do not all act in one plane can usually be reduced to a system, first in one plane, and then in an intersecting plane, and both systems then solved.

The recommended procedure is:

Draw or sketch a **force diagram.**

Select a **free body**, or a **point**, on which the forces to be determined act.

Consider all the forces, known and unknown, on the point or free body selected.

Select the equations, or other method which best applies to the problem, and solve by mathematical or graphical process, as desired. It is often desirable to solve mathematically, and check the results graphically.

Trusses. First solve the system of external forces to determine the reactions, then solve each joint in turn, starting at one end with the known reaction. Both live and dead loads are distributed so as to act at the joints.

Another method is to pass a section through the truss, cutting not more than three unknown members, and making one part of the truss a free body acted on by the external forces and by the stresses in the members cut by the section, and then applying the three equations.

A graphical solution consists in drawing the force polygon for each joint. These polygons may be grouped into one stress diagram.

A graphical method of solving for the reactions by the principle of the equilibrium polygon is given in Art. 50.

CHAPTER VIII

UNIFORM MOTION

This chapter, and the following one, treat of motion which is the second main division of mechanics. The forces which cause or accompany the motion of bodies will be treated in later chapters.

The study of motion includes the meaning of the terms displacement, velocity, and speed ratio, as applied to motion both in a straight line and in rotation. It involves the relations between distance moved and time.

At this point the student should review the brief discussion of motion and velocity in Chapter I, Art. 10; also the application of graphical methods and the parallelogram law to velocities in Chapter II. It should be understood that velocities and accelerations are vector quantities and can be combined or resolved into components in the same manner as forces.

51. Motion. Motion is the change of position of a body with reference to some other body. All motion is relative. A boy in a boat on the river may row upstream against the current and still not progress with respect to the shore.

Suppose a man in the cabin of an airplane flying against the wind (Fig. 135) moves across to the opposite side; he has one direction of motion with respect to the plane, a different motion with respect to the air in which the plane is moving, and a still different motion with respect to the ground below.

A revolving motor pulley has motion with respect to the motor frame; if it is on a moving train it has still another motion with respect to the track.

Motion of translation is motion in which each point in the body passes over a path equal and parallel to the path of every other point; every straight line in the body remains parallel to its original position. It is also called **linear motion.**

Motion of rotation is such that every point in the body travels in a circle about an axis through two fixed points. It is also called **angular motion.**

Frequently a body may have both motion of translation and rotation at the same time. The book in Fig. 12 has motion of

translation; so has the piston of an engine, the crosshead of a locomotive, etc. A revolving flywheel on a stationary engine has motion of rotation. One end of a connecting rod has motion of translation and the other has rotation. The drivers on the moving locomotive have both types of motion. In general, a body may be moved from one position to any given second position in one of three ways: (*a*) by translation in a stated direction; (*b*)

Courtesy Boeing Aircraft Company.

FIG. 135. Modern high-speed, twin-engined transport plane in flight.

by rotation about a given axis; or (*c*) by translation a certain distance in a given line and then rotation through a definite angle about a selected axis.

Consider the case of the slowly moving locomotive in Fig. 136. Any point on the driving wheels has motion of rotation about the axle shaft. Since the locomotive is moving, *all* points are subject to motion of translation; the motion of a point at the center of the axle is the same as that of the locomotive. At any *instant* the point at the top of the driver is moving twice as fast as the center point because it has *both* the motion of rotation and of translation. The point at the bottom where it touches the rail is,

for the instant, stationary with respect to the rail because its forward motion of translation just offsets its motion of rotation. The photographer was successful in catching these relative motions. Note that the stationary point on the rail is in clear focus even to the rivet head, and that the tops and upper parts of the drivers are blurred, even more so than the locomotive itself.

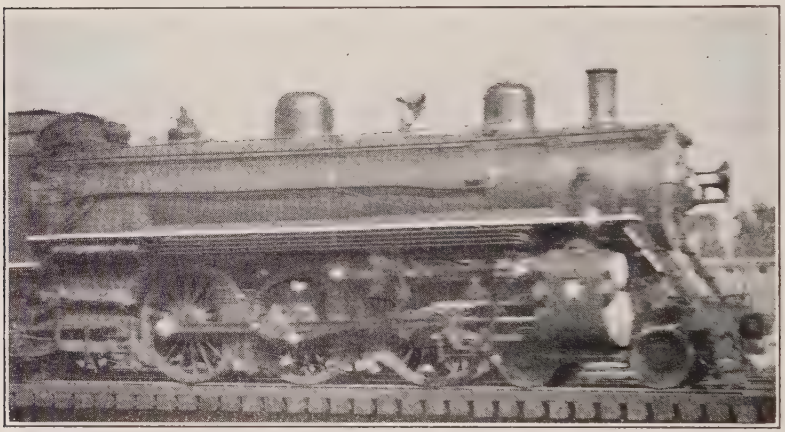


FIG. 136. Locomotive driving wheels have motion of both translation and rotation. The point of the wheel in contact with the rail is in *clear* focus; the tops of the drivers are blurred, more so than the engine itself.

52. Speed. Velocity. The rate of motion of a body is termed its *speed*.

Thus a person may travel with a *speed* of 30 miles per hour, a railroad train may increase its *speed*, etc.

The term **speed** expresses only *two* ideas; *distance* passed over, and *time*. It will be evident, from a little consideration, however, that these alone do not furnish sufficient data for a full consideration of a moving body. Thus, two trains, one running on a straight track, the other around a curve, may have the same speed; the forces to which the two are subject, and the resistance each offers to its motor, will be very different. To account for these forces, we must know, not only the speed, but also the *direction* of the motion, and even the rate at which the direction is changing, as well as the rate at which the speed is changing.

The term **velocity** is applied to a full description of a motion, which states the *three* specifications, *distance*, *time*, and *direction*.

Displacement is the distance moved and the direction of the motion.

Velocity is the rate of displacement.

This distinction between the terms speed and velocity will be made evident by the following illustration: Suppose an airplane starts from a given airport and travels for three hours in a straight line at a *speed* of 100 miles per hour. From this description of the plane's motion, all we know regarding its position at the end of the three hours is that it is somewhere 300 miles away on a straight line from the airport. But suppose the plane moves with a *velocity* of 100 miles per hour *due north.* Its position at the end of the three hours, or after the lapse of any part of the time, is now definitely fixed. A knowledge of the velocity of a body therefore enables us to determine its position after any time. Hence **velocity may be defined as rate of change of position.**

In speaking of the motion of bodies whose direction of motion is definitely recognized as fixed by the conditions, as, for example, a falling body, a part of a machine, a train on a given stretch of track, etc., we may use the terms speed or velocity without danger of confusion, although no actual specification of direction is given for the latter.

The velocity of a body defines its motion at *any given instant* only. Velocity may remain constant in speed and direction, or either may change from instant to instant. Velocity is expressed by stating the distance that the body *would move* in a given (or recognized) direction in one unit of time, if its instantaneous speed and direction *remained unchanged.* The units most commonly used are miles per hour, feet per second (or per minute), revolutions per minute. These units are used even though the motion may not continue for the full distance stated, nor for the full unit of time. Thus an automobile speed of 25 mph does not mean that the car runs the whole hour.

53. Velocity and distance. If the velocity of a body is constant, second after second without variation, the body is said to have uniform velocity or uniform motion.

If the velocity is not constant, the body has variable velocity or motion. Variable velocity is measured at any instant by the instantaneous velocity, i.e., the velocity expressed as though it would continue uniform at that rate. Some examples of variable velocity are the piston of an engine, moving alternately faster and

slower; an automobile starting from rest with constantly increasing speed; a falling stone, etc.

By definition velocity is expressed as displacement (distance and direction) per unit of time. Therefore, the relation between uniform linear velocity and distance is:

$$\textbf{velocity} = \textbf{distance} \div \textbf{time}$$

and

$$\text{distance} = \text{velocity} \times \text{time}$$

Letting s represent distance, v velocity, and t time, the equations are:

$$v = \frac{s}{t} \quad \text{and} \quad s = vt \tag{4}$$

Example 1.

Find the velocity in feet per second of a body which travels 1200 ft in 2 min.

Solution.

$$v = \frac{s}{t} = \frac{1200}{2 \times 60} = 10 \text{ fps} \quad \textit{Ans.}$$

Example 2.

Find the displacement of a machine part which moves to the right for 3 sec, with a velocity of 11 in. per sec.

Solution.

$$s = vt = 11 \times 3 = 33 \text{ in. to the right.} \quad \textit{Ans.}$$

Example 3.

How long will it take to raise a bucket 70 ft if the hoisting rope has a velocity of 5 fps?

Solution.

$$s = vt \qquad 70 = 5 \times t \qquad t = \tfrac{70}{5} = 14 \text{ sec} \quad \textit{Ans.}$$

PROBLEMS

8–1. (*a*) An automobile traveling at 40 mph meets another car traveling in the opposite direction at 50 mph; what is the speed of car number 1 *relative* to car number 2?

(*b*) If car 1 overtakes and passes a third car traveling at 30 mph, what is the speed of car 1 relative to car 3?

(*c*) If a fourth car traveling at 50 mph overtakes and passes car 1, what is the speed of car 1 relative to car 4?

8–2. If the air speed of a plane is 85 mph, what is its ground speed (*a*) in still air? (*b*) in a wind of 30-mph velocity in the same direction? (*c*) in a 20-mph wind in the opposite direction?

8–3. What is the air speed of a plane having a ground speed of 100 mph with a wind in the same direction of 20 mph?

8–4. Find the time required to fly between two airports 400 miles apart, if the ship's air speed averages 80 mph and the wind is blowing (*a*) in the same direction at an average of 25 mph; (*b*) in the opposite direction at 35 mph.

8–5. Find the time required in the preceding problem (*a*) if there is a cross wind of 25 mph; (*b*) if there is a wind of 30 mph at 30° with the line between the two airports.

8–6. A plane made a due east flight of 100 miles in 100 min. If the wind was steady from the southeast at 20 mph, at an angle of 45°, what was the ship's average air speed?

8–7. The velocity of light is 186,400 mps. How long does it take light to come from the sun, distant 92,000,000 miles?

8–8. If the velocity of sound is 1100 fps, how far distant is a gun whose report is heard $3\frac{1}{2}$ sec after the flash is seen?

8–9. Find the time required for a machine part to move a distance of 21 in. if its velocity is 3.3 fps.

8–10. What is the speed in feet per second of an automobile traveling at 40 mph?

8–11. The *reaction time* of an average automobile driver, that is, the time required to see a danger and move one's foot from the accelerator to the brake pedal, is said to be $\frac{3}{4}$ sec. How many feet does a car traveling at 50 mph move in that time?

8–12. The average spacing of a line of telephone poles is 80 ft. At 60 mph how long will it take to travel from one pole to the next?

54. Velocity of rotation. Angular velocity. The velocity of a point on a rotating body, as, for example, a point on the rim of a flywheel, may be expressed in two ways: (1) by stating the linear velocity, that is, the actual length of arc swept over by the point in unit time; or (2) by stating the rate at which the point is changing its direction from the axis of rotation. For example, in a half turn the point can move from directly above to directly below the center of the wheel. The second expression is called the **angular velocity.**

It is evident that the first expression — i.e., the actual linear velocity of the point — depends not only upon how fast the wheel is turning, but also upon *the distance from the point to the axis.* Points farther out from the axis have a greater linear velocity than points situated nearer the center. The angular velocity is, however, the same for all points on the wheel. Angular velocity may be stated, therefore, for the rotating body as a whole.

Angular velocity may be expressed in terms of revolutions per minute, of degrees turned through per unit time, or in terms of a special unit called a **radian**. A radian is the angle θ at the center, which intercepts an arc AB, equal in length to the radius r of the circle, Fig. 137. Since the circumference of a circle equals $2\pi r$, and each radian represents an arc of length r, it is evident that there are $2\pi r/r = 2\pi$ radians in a complete circumference; and that one radian $= 360/2\pi = 57.3°$ approximately.

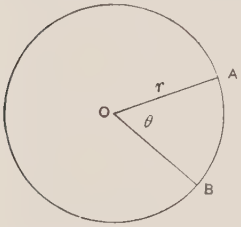


FIG. 137. Arc $AB = r$; angle θ = one radian.

The advantage of this method of expressing angular velocities is evident from the fact that

The angular velocity in radians times the distance r from a point to the axis of rotation equals the linear velocity of the point.

Letting v represent the linear velocity of the point in feet per second, r the radius in feet, and ω (the Greek letter omega) the angular velocity in radians per second, the relation is given by the equations:

$$v = \omega r \quad \text{and} \quad \omega = \frac{v}{r} \tag{5}$$

To express the relations entirely in angular units, let θ (the Greek letter theta) represent the angular displacement, i.e., the number of radians (or degrees or revolutions) swept over, then

$$\omega = \frac{\theta}{t} \quad \text{and} \quad \theta = \omega t \tag{4a}$$

These equations are similar in form to equation (4) in Art. 53.

In connection with machinery, velocity of rotation is commonly expressed in revolutions per minute. From the fact that a complete circumference equals 2π radians it will be seen that

$$1 \text{ rpm} = 2\pi \text{ radians per min}$$

Example 1.

If a wheel has a velocity of 5 radians per sec, find the linear velocity of a point on the wheel 2 ft from the axis.

Solution.

$$v = \omega r = 5 \times 2 = 10 \text{ fps} \quad \textit{Ans.}$$

Example 2.

A pulley is revolving at 1000 rpm; find its velocity in radians per minute.

Solution.

Radians = rpm $\times 2\pi$ = $1000 \times 2 \times 3.14$ = 6280 radians per min. *Ans.*

Example 3.

Find the linear velocity of a point on the rim of a 10 in. diam. pulley running at 1000 rpm.

Solution.

$$v = \omega r = \frac{1000}{60} \times 2\pi \times \frac{5}{12} = 43.6 \text{ fps} \quad \textit{Ans.}$$

Example 4.

Find the angular velocity in radians per second, if the total radians turned is 3000 and the time is 2 min.

Solution.

$$\omega = \frac{\theta}{t} = \frac{3000}{2 \times 60} = 25 \text{ radians per sec} \quad \textit{Ans.}$$

55. Motion of machine parts. Gears. *Gear trains* are used for transmitting motion for short distances, and for large reduction or increase in speed. Gears have the advantage of positive action, without slip as found in belt and friction drives.

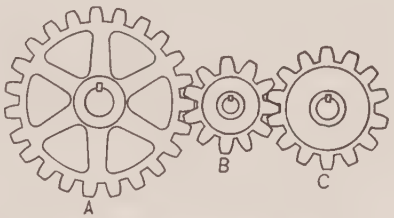


FIG. 138. Simple gear train.

A *simple* train of spur gears, as in Figs. 138 and 139, is one in which each gear meshes with the adjoining one; each gear being a "driver" of the one following and a "follower" of the one preceding. A *compound* train is made up of two or more simple trains, as in Fig. 140, in which the last follower of the first train is keyed to the same shaft as the driver of the second train, as B and C on shaft N.

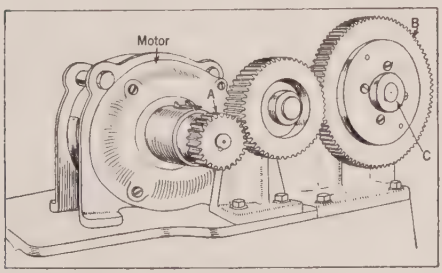


FIG. 139.

A study of the diagrams, or of a set of gears, will make clear the following statements regarding a simple train. The "pitch" of the teeth on each gear, i.e., the number per inch of circumference, must be the same. Any two gears in mesh rotate in opposite directions; if it is desired that two shafts shall turn in the same direction, an "idler" must be placed between the two gears attached to the shafts. As a driven gear, or follower, must rotate one tooth for each tooth on the driver, the revolutions per minute of the driving and driven shaft are inversely proportional to the numbers of teeth on their gears.

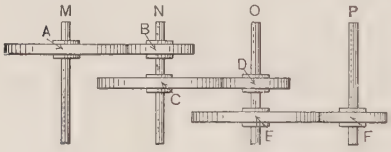


FIG. 140. Compound gear train.

The *speed* of a gear usually means its angular speed in revolutions per minute. The *tooth speed* of a gear means the number of teeth per minute passing any point, and equals its number of teeth times its revolutions per minute. The tooth speed of all gears in a simple train is the same, since all are in mesh. Therefore the following equation is true: The revolutions per minute of any gear times its number of teeth equals the revolutions per minute of any other gear in the same simple train times its number of teeth.

From this relation the speed of any gear in a train may be determined, or the size of a gear to give a desired speed. Let t_1 represent the teeth in one gear and t_2 the teeth in any other, also rpm_1 and rpm_2 the respective speeds.

Then $$\text{rpm}_1 \times t_1 = \text{rpm}_2 \times t_2 = \text{rpm}_3 \times t_3, \text{ etc.} \quad (6)$$

from which $$\text{rpm}_2 = \frac{\text{rpm}_1 \times t_1}{t_2} \quad \text{and} \quad t_2 = \frac{\text{rpm}_1 \times t_1}{\text{rpm}_2}$$

A compound train can be figured by taking each simple part separately. Or, referring to Fig. 140, the following equations may be developed:

$$\text{rpm of shaft } N = \frac{\text{no. of teeth on } A}{\text{no. of teeth on } B} \times \text{rpm of shaft } M$$

$$\text{rpm of shaft } O = \frac{\text{no. of teeth on } C}{\text{no. of teeth on } D} \times \text{rpm of shaft } N$$

$$\text{rpm of shaft } P = \frac{\text{no. of teeth on } E}{\text{no. of teeth on } F} \times \text{rpm of shaft } O$$

Therefore, rpm of shaft P

$$= \frac{\text{no. teeth on } A \times \text{no. teeth on } C \times \text{no. teeth on } E}{\text{no. teeth on } B \times \text{no. teeth on } D \times \text{no. teeth on } F} \times \text{rpm}$$

of shaft M;

or

The rpm of the driven shaft

$$= \frac{\text{the product of the no. of teeth on all the drivers}}{\text{the product of the no. of teeth on all the driven wheels}} \times$$

rpm of the driving shaft. Notice that the driver of each train has the *same angular speed* as the follower of the preceding train on the same shaft, but has a *different tooth speed.*

Bevel gears are used on shafts at right angles to each other. They conform to the same speed relations as do spur gears. *Worm* or *screw* gears have a worm or screw-like thread on one shaft in mesh with a gear wheel having teeth shaped to fit on the other shaft which is at right angles to the first. One revolution of the gear requires one turn of the worm shaft for each tooth on the gear, if the worm is single thread; and half as many turns if the thread is double.

Example.

Find the speed and direction, also the tooth speed, of the third gear in a simple train, if it has 32 teeth and the first gear has 48 teeth and rotates clockwise at 40 rpm.

Solution. The tooth speed is the same for all gears; it is equal to number of teeth $\times$ rpm $= 48 \times 40 = 1920$ teeth per min.

$$\text{rpm}_3 = \frac{\text{rpm}_1 \times t_1}{t_3} = \frac{48 \times 40}{32} = 60 \text{ rpm} \quad Ans.$$

Direction of third gear = same as first (opposite to second) = clockwise.

Figure 141 represents the arrangement of automobile transmission gears for the *standard shift.* When running, the motor turns gear A which drives gear D on the countershaft. All gears on the countershaft have the same speed. When in *neutral* the drive shaft has no motion from the motor, as the gears are not in mesh and the motor shaft and drive shaft are separated between A and B, except for a slip joint. When the operator

moves the gear shift lever toward his knee and pulls backward, the sliding gear C is moved forward into mesh with F giving *first* speed on the drive shaft. When gear shift lever is moved forward,

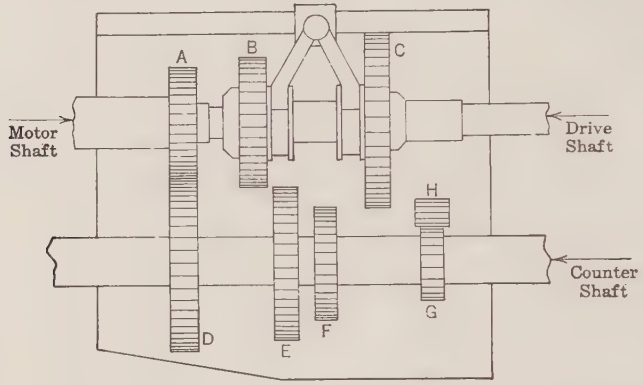


FIG. 141. Automobile transmission gears in neutral position.

gear C is moved backward, out of mesh with F and into mesh with pinion H driven by G giving *reverse*. When the driver moves the gear shift lever away from his knee, gear C is left in neutral and gear B is engaged, and can be slid backward into mesh with E for *second* speed, or can be slid forward and *clutched* with A by an arrangement of spur teeth on the beveled side of the gear, giving the same speed as A. This gives the *direct drive* or *third* speed, in which case the countershaft merely idles.

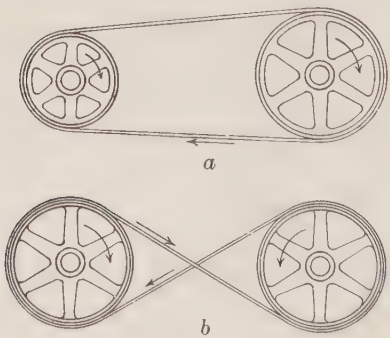


FIG. 142. Belt drives.

56. Belt drives. Where two shafts are parallel and some distance apart, a common method of transmitting motion from one to the other is to use a belt running on pulleys fixed to the two shafts as in Fig. 142. If an open belt is used, as in Fig. 142(a), the two shafts will rotate in the same direction; if a crossed belt is used, as in Fig. 142(b), the two shafts will rotate in opposite directions.

The *belt speed* equals the rim speed of the pulley, that is the

linear speed of the face of the pulley. The rim speed equals the rpm $\times$ the circumference, i.e.,

$$\text{Rim speed} = \text{rpm} \times D \times \pi \tag{7}$$

When there is no belt slip, the linear speed of the pulley faces must be the same; hence if r_A is the radius of the belt pulley on shaft A and r_B the radius of the belt pulley on shaft B, $2\,\pi r_A \times$ rpm of shaft $A = 2\,\pi r_B \times$ rpm of shaft B. From this equation, it is apparent that the shaft speeds are inversely proportional to the radii of the belt pulleys.

Because the circumference is proportional to the diameter the above relation can be stated as follows: Let D_1 represent one diameter and D_2 the other; also let rpm_1 and rpm_2 be the respective angular velocities.

Then
$$\text{rpm}_1 \times D_1 = \text{rpm}_2 \times D_2 \tag{8}$$

From which

$$\text{rpm}_2 = \frac{\text{rpm}_1 \times D_1}{D_2} \quad \text{and} \quad D_2 = \frac{\text{rpm}_1 \times D_1}{\text{rpm}_2}$$

A more accurate check of actual conditions will be obtained by considering the belt speed as average at the center of the thickness of the belt, and this can be accomplished in the foregoing equations *by adding the belt thickness to each pulley diameter* (i.e., one-half the thickness on each face of the pulley).

In any actual case of power transmission by a belt, there is some slipping of the belt, the amount of which depends on the power transmitted and the tightness of the belt. The actual speeds of the shafts may be found by counting each with the aid of a revolution counter, Fig. 143. A comparison of the measured speed of the driven shaft with its speed computed from the speed of the drive shaft and the size of the belt pulleys will give the revolutions lost through the slipping of the belt. Thus suppose the computed speed of the driven shaft is 320 rpm and the measured speed 304 rpm; there is a loss of 16 rpm or $\frac{16}{320} \times 100 = 5$ per cent of the motion

FIG. 143. Revolution counter.

(and therefore of the power) which might have been transmitted, because of the slipping of the belt.

Example 1.

Find the belt speed in fpm when the driving pulley is of 5 in. diam and is making 1000 rpm, assuming no belt slip, and neglecting belt thickness.

Solution.

$$\text{Belt speed} = \text{circumference} \times \text{rpm}$$

$$= \frac{3.14 \times 5 \text{ in.} \times 1000}{12} = 1309 \text{ fpm} \quad \textit{Ans.}$$

Example 2.

Find the speed of a 20-in. diameter pulley driven by the pulley and belt of Example 1.

Solution.

$$\text{rpm}_2 = \frac{\text{rpm}_1 \times d_1}{d_2} = \frac{1000 \times 5}{20} = 250 \text{ rpm} \quad \textit{Ans.}$$

Example 3.

Find the speed of the foregoing 20-in. pulley if the belt thickness of $\frac{1}{4}$ in. is considered.

Solution.

$$\text{rpm}_2 = \frac{\text{rpm}_1 \times d_1}{d_2} = \frac{1000 \times 5.25}{20.25} = 259+ \text{ rpm} \quad \textit{Ans.}$$

PROBLEMS

8–13. The drive wheels of a locomotive are 18 ft in circumference. How many revolutions per minute will these wheels be making when the locomotive is moving at the rate of 50 mph?

8–14. Which is the greater velocity, 100 rpm or 10 radians per sec, and by how much?

8–15. A wheel is rotating with a uniform speed of 45° per sec. How many revolutions will it make in half an hour? Express its velocity in revolutions per minute. In radians per second.

8–16. Through what distance will a point on the wheel of Problem 8–15, 2 ft from the axis, move in 5 min?

8–17. A wheel is rotating with a velocity of 2 radians per sec. Through what distance will a point on the wheel 18 in. from the axis move in 1 min?

8–18. Express the velocity of rotation of the minute hand of a clock in degrees per second. In radians per second.

8–19. In a simple-gear train, gear (1) has 64 teeth, (2) has 32 teeth, (3) has 40 teeth, (4) has 16 teeth, (5) has 80 teeth. If gear (1) rotates counterclockwise at 80 rpm, what is the direction and speed of each gear? What is the tooth speed of each gear?

8–20. To drive a shaft C in Fig. 139 at 600 rpm a simple train of three gears was used, gear A having 48 teeth and making 1500 rpm. How many teeth were required to be in gear B on shaft C?

8–21. In an arrangement of gears as in Fig. 140, gear A has 84 teeth, gear B 14, gear C 80, gear D 16, gear E 54, and gear F 18. If shaft M is turned at a speed of 40 rpm, what will be the speed of shaft P?

8–22. In a compound gear train, A with 68 teeth drives B with 17 teeth, which drives C with 34 teeth; D with 62 teeth is keyed to the same shaft as C and drives E with 31 teeth. Find speed and direction of each gear, if A revolves clockwise at 22 rpm.

8–23. In a worm and worm gear like Fig. 144, the wheel has 98 teeth. If the worm shaft is turning at 1600 rpm, what is the speed of the gear?

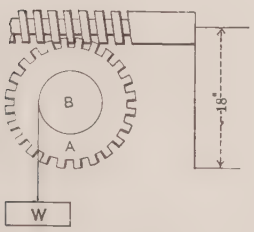


Fig. 144.

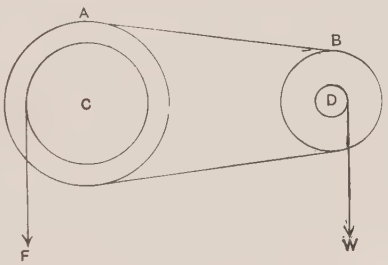


Fig. 145.

8–24. A 12-in. pulley on an engine shaft makes 210 rpm and drives by belt a 30-in. pulley on a second shaft. This shaft carries a worm which is in mesh with a gear on a third shaft. If this gear has 108 teeth, compute the speed of the third shaft.

8–25. The diameter of pulley A, Fig. 145, is 18 in.; of pulley B is 12 in.; of drum C 14 in.; of drum D 4 in. Find the distance moved by F when W is raised 1 ft.

8–26. A motor pulley $4\frac{1}{2}$ in. in diameter drives a 14-in. machine pulley by a belt $\frac{1}{4}$ in. thick. If the motor speed is 900 rpm, find the speed of the machine pulley. Find the speed if there is a belt slip of 10 per cent.

8–27. What sized pulley must be used on a motor having a speed of 1800 rpm to give a speed of 1100 rpm to an 8-in. diameter lathe pulley?

8–28. A motor with a 5-in. pulley running at 1500 rpm drives a countershaft with a 12-in. pulley. A 10-in. pulley on this shaft drives a wood-lathe having a 4-in. pulley. How fast will the lathe turn?

8–29. What sized pulley will be needed on a machine to give it a speed of 1500 rpm if driven by a 10-in. pulley on a countershaft which is driven through a 12-in. pulley from a 6-in. pulley on a motor running at 1000 rpm?

8–30. Workmen moved a heavy piece of machinery by placing rollers of 4-in. diameter under it. How far forward along the floor would it move for each revolution of the rolls?

8–31. In Problem 8–13 what is the linear speed in feet per second, *at any instant* (*a*) of the locomotive, relative to the track? (*b*) of the point on the driver in contact with the rail, relative to the track? (*c*) of a point at the top of the driver, relative to the track? (*d*) of a point at the top of the driver, relative to the frame of the engine?

8–32. In an arrangement of gears as in Fig. 200, *A* has 81 teeth, *B* 60 teeth, *C* 36 teeth. The drum for *F* is 9 in. in diameter, for *W* is 6 in. in diameter. Find the distance moved by *F* when *W* is lifted 1 ft.

8–33. Referring to the automobile transmission gears in Fig. 141, the number of teeth in each gear is as follows: *A* 12, *B* 19, *C* 24, *D* 29, *E* 22, *F* 14, *G* 11, and *H* 16. (*a*) Find the ratio of the motor speed to the drive-shaft speed for each case, when using first, second, third, and reverse speeds. (*b*) If the speed of the motor is 1650 rpm, find the drive-shaft speed for each case.

SUMMARY

Motion is the change of position of a body relative to some other body. Motion of **translation** is motion in which each point in the body passes over a path equal and parallel to the path of every other point, and every straight line in the body remains parallel to its original position. It is also called **linear motion.** Motion of **rotation** is motion in which every point in the body travels in a circle about an axis through two fixed points. It is also called **angular motion.**

Speed is the rate of motion of a body. It equals distance divided by time.

Velocity is the rate of motion of a body in a given direction. It equals displacement divided by time.

Displacement is the distance moved in a stated direction. Displacement and velocity are **vector** quantities. The terms speed and distance can be understood to mean velocity and displacement, respectively, when referred to machines or to conditions in which the direction is known. The equations are:

$$v = \frac{s}{t} \quad \text{and} \quad s = vt \tag{4}$$

Angular displacement is the angle and direction swept through, expressed in degrees, in whole revolutions, or in radians.

Angular velocity, that is, velocity of **rotation**, is expressed in terms of the length of arc per unit of time (i.e., linear velocity on the arc); or in terms of the angle swept through per unit of time.

A radian is the angle which intercepts an arc equal to the radius. 2π radians = 360 degrees.

$$\omega = \frac{\theta}{t} \quad \text{and} \quad \theta = \omega t \tag{4a}$$

The equations giving the relation between linear and angular velocity are:

$$v = \omega r \quad \text{and} \quad \omega = \frac{v}{r} \tag{5}$$

The speeds of machine parts are expressed either in linear or in angular terms.

A simple gear train is one in which each gear is in mesh with one preceding or following. A **compound train** is made up of two or more simple trains. Tooth speed is the linear rim speed of a gear in number of teeth passing a point per unit of time. Tooth speed = rpm $\times$ t. The tooth speed of all gears in a simple train is the same. The equation for simple gears is:

$$\text{rpm}_1 \times t_1 = \text{rpm}_2 \times t_2 = \text{rpm}_3 \times t_3, \text{ etc.} \tag{6}$$

Belt speed is the same as the rim speed of both pulleys if there is no belt slip. It is customary to compute the speeds without slip, and then allow a percentage for slipping.

$$\text{Belt speed} = \text{rpm} \times D \times \pi \tag{7}$$

The equation for pulley speed (angular) is:

$$\text{rpm}_1 \times D_1 = \text{rpm}_2 \times D_2 \tag{8}$$

SYMBOLS FOR UNIFORM MOTION AND SPEEDS

D = diameter.
r = radius.
rpm = revolutions per minute.
s = distance; displacement.
t = time.
v = velocity; (speed).
θ = angular displacement.
ω = angular velocity.

CHAPTER IX

ACCELERATED MOTION

The motion of a body is seldom uniform. Changes in velocity constantly occur, as when an automobile is accelerated in starting, or retarded in stopping, or when its speed of motion is increased or decreased. Chapter IX treats of these changes in velocity. The equations for linear acceleration as well as angular acceleration are explained. Falling bodies and projectiles are included, as is also the particular type of accelerated motion known as simple harmonic motion. An article on the cam and also the simple linkage types of motion in machines is also included.

57. Acceleration. The rate of change in velocity per unit of time is called **acceleration.** When the change of velocity occurs uniformly, either increasing or decreasing, the motion is called *uniformly accelerated motion.* A train starting from rest at a station, and gradually getting under headway, is an illustration of accelerated motion. If the force starting the train is uniform and the resistance to motion constant, the motion of the train will be *uniformly* accelerated. The same train pulling into a station and gradually stopping under the application of the brakes is an illustration of accelerated motion where the velocity is decreasing, i.e., of *negatively* accelerated or *retarded* motion.

Suppose a body starts from rest under the influence of a constant force, having a velocity of zero (at rest) at the start; a velocity of 10 fps at the end of the first second; 20 fps at the end of the next second; 30 fps at the end of the third second, etc. The gain in velocity is evidently 10 fps in each second, i.e., 10 ft per sec per sec. This is commonly written 10 ft per sec^2 or 10 fps^2. The total change in velocity is 30 fps.

The relation between acceleration, change in velocity, and time evidently is:

$$\textbf{acceleration} = \textbf{change in velocity} \div \textbf{time}$$

and

$$\text{change in velocity} = \text{acceleration} \times \text{time}$$

Letting a represent linear acceleration and v_c the velocity change, the equations become:

$$a = \frac{v_c}{t} \quad \text{and} \quad v_c = at \tag{9}$$

In the foregoing case the acceleration is considered positive because it increases the velocity, whereas a retardation of velocity is considered a negative acceleration. In either case the above equation is true if the proper algebraic sign is used. The acceleration is always in the direction of the force acting to produce change of velocity, whether in the direction of the original motion or opposed to it.

Example.

Find the change in velocity of a body if it is *retarded* at the rate of 10 fps^2 for 3 sec.

Solution. Acceleration here is negative $= -10$ fps^2

$$v_c = a \times t = -10 \times 3 = -30 \text{ fps} \quad \textit{Ans.}$$

That is, the velocity is 30 fps less at the end of 3 sec. If its original velocity had been 30 fps, the body would now be at rest.

58. Acceleration of rotating bodies. Angular acceleration. Acceleration in motion of rotation may be expressed in the two ways already stated for the velocity. Thus, we may state the *change in linear velocity per unit of time*, i.e., **the linear acceleration** of a point at a known distance from the axis of rotation, or we may state the *change in angular velocity per unit of time*, i.e., the **angular acceleration.** The linear acceleration of different points in a rotating body whose velocity is changing evidently depends upon the distances of the points from the axis; the angular acceleration is the same for all points. Angular acceleration may be expressed in revolutions per unit of time gained or lost each unit (e.g., 10 rpm each sec), or as radians per second gained or lost each second.

It is evident that, just as the angular velocity of a point expressed in radians $\times$ radius = linear velocity of the point, so also:

Angular acceleration in radians per sec^2 $\times$ radius in ft = linear acceleration in ft per sec.^{2}

Let α (the Greek letter alpha) represent angular acceleration in radians per sec.^{2}

Then $$a = \alpha r \tag{10}$$

and $$\alpha = \frac{\omega}{t} \qquad (9a)$$

which is similar to equation (9) for linear acceleration, in Art. 57.

Example.

A wheel making 200 radians per sec has its velocity increased in 15 sec to 305 radians per sec. Find the acceleration.

Solution. The change in velocity = 305 − 200 = 105 radians per sec. $\alpha = \frac{\omega}{t} = \frac{105}{15} = 7$ radians per sec², acceleration. *Ans.*

PROBLEMS

9–1. A velocity change from 0 fps to 60 fps occurs in 10 sec. Find the acceleration.

9–2. If the acceleration is 7 fps², and the initial velocity is 6 fps what will be the velocity at the end of 8 sec?

9–3. If at 1 o'clock the velocity of a train is 15 mph and at 2 min past 1 is 20 mph, what is the acceleration in miles per hour per minute?

9–4. What is the acceleration in Problem 9–3, expressed in feet per second per second?

9–5. At 2 o'clock the train of Problem 9–3 is traveling with a velocity of 40 mph. With a negative acceleration of 0.5 fps² at what time will it come to a stop?

9–6. A certain body moves with an acceleration of 80 fps². What will be the change in velocity in half a minute?

9–7. Express the acceleration in Problem 9–6 in feet per minute per second.

9–8. If the initial velocity of a body is 1000 fps due north, and its acceleration is 50 fps² due south, find the velocity of the body at the end of 25 sec.

9–9. The velocity of a body has changed from 8 fps due east to 208 fps due west in 6 sec. What has been its acceleration?

9–10. A wheel making 120 rpm has its velocity increased in 10 sec to 180 rpm. Find the acceleration: (*a*) in revolutions per minute gained each second; (*b*) in radians per second per second.

9–11. A point on a rotating body is moving 50 fps. The angular velocity of the body is increased 1 rpm each second for 10 sec. What will then be the velocity of the point in feet per second if it is situated 5 ft from the axis of rotation?

9–12. A wheel has a velocity of 1200 rpm. If the speed is reduced one-half in a minute, find the acceleration (*a*) in revolution per minute per second; (*b*) in radians per second per second.

59. Uniformly accelerated linear motion. Average velocity and distance. In the preceding articles it was developed that change in velocity equals acceleration times time. It is also evi-

dent that at the end of any given time **final velocity equals initial velocity plus change in velocity.** Letting v_f represent final velocity, and v_0 initial velocity (at zero time), this relation can be expressed:

$$v_f = v_0 + v_c \quad \text{also} \quad v_f = v_0 + at \tag{11}$$

the second form of the foregoing equation being obtained by substitution from equation (9).

The average velocity for a given period of time will be one-half the final velocity if the body starts from rest, or for the general case of the body already in motion, **average velocity equals one-half the sum of the initial and final velocities.** Letting v_a represent the average velocity, the equation will be:

$$v_a = \frac{v_0 + v_f}{2} \quad \text{also} \quad v_a = v_0 + \tfrac{1}{2}\,at \tag{12}$$

The second form of equation (12) is derived thus:

$$v_a = \frac{v_0 + v_f}{2} = \frac{v_0 + (v_0 + at)}{2} = \frac{2\,v_0 + at}{2} = v_0 + \tfrac{1}{2}\,at$$

If a body moves with a uniform acceleration, the distance that it will move in a given time can be determined by using the average velocity for that time. If the body should move for the given time with uniform velocity equal to the average, the distance would be: $s = vt$, equation (4). Therefore, **distance = average velocity × time.** The equation will be

$$s = v_a t \tag{4b}$$

Equation (4*b*) is another form of equation (4). By substitution from equation (12) the distance equation becomes:

$$s = (v_0 + \tfrac{1}{2}\,at)\,t = v_0 t + \tfrac{1}{2}\,at^2 \tag{13}$$

Example 1.

Suppose a body to move with a uniform acceleration and let its velocity at the time we begin to observe its motion (initial velocity) be 6 fps. At the end of 5 sec suppose the velocity to have become 26 fps. Find acceleration and distance.

Solution.

$$\text{Change of velocity} = 26 - 6 = 20 \text{ fps}$$

$$\text{Acceleration} = \text{rate of change of velocity} = \frac{26 - 6}{5} = \frac{20}{5} = 4 \text{ fps}^2 \quad \textit{Ans.}$$

The *average velocity* during the 5 sec will be *one-half the sum of the initial and the final velocities*, or

$$\text{Average velocity} = \frac{6 + 26}{2} = 16 \text{ fps}$$

The total distance traversed will be the average velocity multiplied by the time, or

$$\text{Distance} = 16 \times 5 = 80 \text{ ft} \quad \textit{Ans.}$$

It is of the utmost importance for the student to master thoroughly these foregoing simple relations. They apply equally to rotation and translation.

It should be noticed that these statements are *perfectly general for all cases* of uniformly accelerated motion. Thus, if a body starts from rest, its initial velocity is 0. Also, if the initial velocity is in one direction and the acceleration is in the *opposite* direction, the rules above still apply if we give to one direction the + sign and to the other the − sign.

Example 2.

A body is moving N. with a velocity of 40 fps and a force acts S. so as to give it an acceleration S. of 8 fps^2. What will be its velocity at the end of 7 sec, and how far from the start will it be at the end of that time?

Solution. Call velocity and acceleration North +, velocity and acceleration South −. The original velocity is here, therefore, +, the acceleration −.

$$\text{Total change of velocity} = -8 \text{ fps} \times 7 = -56 \text{ fps}$$

$$\text{Final velocity} = +40 \text{ fps} - 56 \text{ fps} = -16 \text{ fps}$$

$$\text{Average velocity} = \frac{40 \text{ fps} + (-16 \text{ fps})}{2} = 12 \text{ fps}$$

$$\text{Distance} \quad 12 \times 7 = 84 \text{ ft}$$

That is, the body will, at the end of the 7 sec, be 84 ft N. of its original position, and will be moving S. with a velocity of 16 fps. *Ans.*

(What has actually happened if the full history of the motion is required, is that the body has moved N. until brought to rest, then has started S. from zero velocity and moved until its velocity is 16 fps S.)

To lose all velocity N. will require $\frac{40}{8} = 5$ sec, and the distance moved N. will be $\frac{40 + 0}{2} \times 5 = 100$ ft.

The time it moved South is therefore, $7 - 5 = 2$ sec. Velocity at the end of the seventh second is, therefore, $2 \times (-8) = -16$ fps S. Distance moved S. after stopping $= \dfrac{0 - 16}{2} \times 2 = -16$ ft. Therefore the body is $100 - 16 = 84$ ft N. of the starting point.

60. Accelerated motion from rest. If a body starts from rest, the foregoing statements and equations hold true, as v_0 will have the value of zero, and then $v_f = v_c$.

From equation (11) $\qquad v_f = at \qquad (11R)$

From equation (12) $\qquad v_a = \frac{1}{2}\, at \qquad (12R)$

From equation (13) $\qquad s = \frac{1}{2}\, at^2 \qquad (13R)$

The *distance* passed over by a body starting from rest and moving with a uniform acceleration, therefore, *varies as the square of the time.* Hence, if the body moves 10 ft the first second, it will move $2 \times 2 \times 10 = 40$ ft in 2 sec, $3 \times 3 \times 10 = 90$ ft in 3 sec, etc.

Experimental data, plotted as in Fig. 146, will conform to the relationships of the above equations, that is:

$$\text{velocity} \propto \text{time}$$

$$\text{distance} \propto (\text{time})^2$$

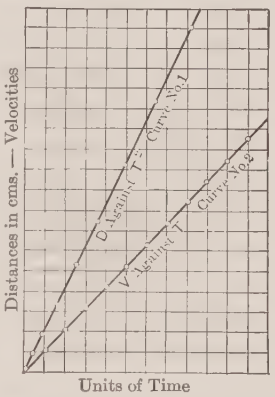


FIG. 146. Relation of velocity and distance, in accelerated motion from rest, to time.

As it is often desired to know the relation between velocity acceleration and distance, without necessarily determining the time, a single equation for this relation can be derived from equations (11R) and (13R). Since this is a case of accelerated motion from rest, let velocity equal v, standing for either v_c or v_f.

Thus from (11R) $\qquad t = \dfrac{v}{a}$

Substituting in (13R) $\qquad s = \dfrac{a}{2} \times \dfrac{v^2}{a^2} = \dfrac{av^2}{2\,a^2} = \dfrac{v^2}{2\,a}$

And transposing: $\qquad v^2 = 2\,as \quad \text{and} \quad v = \sqrt{2\,as}$

That is $\qquad s = \dfrac{v^2}{2\,a} \quad \text{and} \quad v = \sqrt{2\,as} \qquad (14R)$

Note that v in this equation represents the velocity acquired by a body starting from rest, with an acceleration a, in a distance s. Note also that for accelerated motion from rest, the relation of *velocity and time* is given by equation $(11R)$; of *time and distance* by $(13R)$; of *velocity and distance* by $(14R)$.

61. Uniformly accelerated circular motion. The principles and procedure for angular acceleration are similar to those for linear acceleration. Since, for a rotating body, the linear distance moved by any selected point on the rotating body is proportional to the angular displacement of the point in the same time, the laws of accelerated motion of rotation under a constant torque are exactly analogous to the laws of accelerated motion in a straight line under the action of a constant force.

As in the case of linear motion the *elementary* relations between velocity, acceleration, distance, and time are sufficient for the solution of most problems. In cases where time will be saved by the use of the equations, they may be adapted for use for angular motion by substituting ω for v, velocity, α for a, acceleration, and θ for s, displacement. The displacement, in the case of circular motion, is expressed in terms of the angle swept over, measured in degrees, in radians or in revolutions ($= 2\pi$ radians). When transferring from angular to linear units, α, ω, and θ should be expressed in radians.

Example 1.

A wheel starting from rest is given an acceleration of 3 radians per sec^2. Through what distance will a point 2 ft from the axis of rotation move in 20 sec?

Solution. Since the radius of the circle in which the point will move is 2 ft, 1 radian equals 2 ft.

Therefore linear acceleration of the point equals

$$3 \times 2 = 6 \text{ fps}^2$$

$$\text{Change in velocity} = 6 \times 20 = 120 \text{ fps}$$

$$\text{Average velocity} = \frac{0 + 120}{2} = 60 \text{ fps}$$

$$\text{Distance moved in 20 sec} = 60 \times 20 = 1200 \text{ ft.} \quad \textit{Ans.}$$

Or by using equations $(13R)$ and (10):

$$s = \tfrac{1}{2} at^2 = \tfrac{1}{2} \alpha t^2 \times r = \tfrac{1}{2} \times 3 \times (20)^2 \times 2 = 1200 \text{ ft} \quad \textit{Ans.}$$

Example 2.

A motor pulley is accelerated from rest until at the end of 1 min it is making 1000 rpm. Find (*a*) its acceleration, (*b*) the number of revolutions during the minute, (*c*) the distance moved by a point $\frac{1}{2}$ ft from the center of the pulley.

Solution.

(*a*) From (9*a*)

$$\alpha = \frac{\omega}{t} = \frac{1000}{1} = 1000 \text{ rpm per min acceleration.} \quad \textit{Ans.}$$

$$\text{In radians, acceleration} = \frac{1000 \times 2\pi}{60 \times 60} = 1.74 \text{ radians per sec}^2$$

$$(b) \qquad \text{Average } \omega = \frac{\omega_0 + \omega_f}{2} = \frac{0 + 1000}{2} = 500 \text{ rpm}$$

From (4*a*)

$$\theta = \omega t = 500 \times 1 = 500 \text{ revolutions displacement in 1 min.} \quad \textit{Ans.}$$

(*c*) Displacement in radians = revolutions $\times 2\pi$

$$= 500 \times 2 \times 3.14 = 3140 \text{ radians}$$

Linear displacement = radians $\times r$

$$= 3140 \times 0.5 = 1570 \text{ ft displacement.} \quad \textit{Ans.}$$

Answer (*b*) can be obtained also by substituting the angular symbols in equation (13*R*) whence:

$$\theta = \tfrac{1}{2}\alpha t^2 = \tfrac{1}{2} \times 1000 \times 1^2 = 500 \text{ rpm, } \textit{as before}$$

Answer (*c*) also can be obtained thus:

From (5) $v = \omega r = (500 \times 2\pi) \times 0.5 = 1570$ fpm, average

From (4) $s = vt = 1570 \times 1 = 1570$ ft, *as before*

62. Falling bodies. A body falling under the influence of gravity, if we neglect the resistance offered by the air, furnishes a common instance of uniformly accelerated motion.

The value of the force of gravity and therefore of the acceleration, g, that it produces varies somewhat from place to place on the earth's surface, being, in general, least at the equator and greater as the poles are approached. It is also greater at the surface of the earth than at points below or above the surface. These variations, however, are small. The approximate value for sea level is about 32.2 fps^2 (in the metric system, 980 cm per sec^2). For the purpose of most problem work, in which altitude need not

be taken into account, an even 32 fps^2 is considered sufficiently exact, and is the value that will be used in this book.

By substituting this value, represented by the symbol g, into equations of the preceding articles, and also substituting h (height) for s, the equations for falling bodies are:

From (13) For bodies having initial velocity $h = v_0 t + \frac{1}{2} gt^2$ (13F)

From (13R) For bodies falling freely from rest $h = \frac{1}{2} gt^2$ (13FR)

From (14R) $h = \dfrac{v^2}{2g}$ and $v = \sqrt{2gh}$ (14FR)

63. Body projected vertically upward or downward. Suppose a ball is projected vertically upward with a velocity of 192 fps; required to find: (*a*) The time during which it will rise; (*b*) the height to which it will rise; (*c*) its velocity after 8 sec; (*d*) its velocity on striking the ground.

(*a*)
Initial velocity = 192 fps
Acceleration = -32 fps^2
Change in velocity = $-32\,t$ fps

Therefore, since its velocity at the highest point of its journey (final velocity) is 0, we have

$$0 = 192 - 32\,t$$

or
$$t = 6 \text{ sec}$$

(*b*)
Initial velocity = 192 fps
Final velocity = 0 fps
Average velocity = 96 fps
Distance = $96 \times 6 = 576$ ft, or body will rise 576 ft

(*c*) Change of velocity in 8 sec = -32 fps $\times$ 8 = -256 fps

Final velocity = $192 + (-256) = -64$ fps

Or body will be moving 64 fps *downward.*

(*d*) It took 6 sec to reach the highest point where the velocity was 0. Considering now only the return journey, we may call the

Initial velocity = 0
Acceleration = 32

Final velocity $= 32\,t$ (where t is the time of the return journey)

Average velocity $= 16\,t$

Distance $= 576$ ft

or $576 = 16\,t \times t = 16\,t^2$

$t = 6$

and the velocity on reaching the ground will be $6 \times 32 = 192$ fps.

Thus we see that, in this case, the time of rise will be equal to the time of the return. The whole time of flight is 12 sec. Also the downward velocity on reaching the ground will be numerically equal to the upward velocity with which the body started.

In the case of a body projected vertically downward we may apply the laws of accelerated motion in the same way, the initial velocity and the acceleration of gravity being here in the same direction.

PROBLEMS

9–13. A sled moves from rest until in 12 sec it has acquired a velocity of 60 fps. Find the acceleration, the average velocity, the velocity at the end of 7 sec, and the distance traversed in the 12 sec.

9–14. If an engine takes 24 sec to get up a velocity of 300 rpm in its flywheel, find: (*a*) the angular acceleration of the flywheel, (*b*) the angular velocity of the flywheel at the end of 15 sec, (*c*) the average angular velocity during the process of speeding up.

9–15. The acceleration of a body starting from rest is 3 fps^2. Plotting times and velocities for the end of each of the first 5 sec, get the time-velocity diagram and draw on it the line representing the average velocity. Determine also the distance. Plot time on the horizontal axis.

9–16. For 6 sec after starting, a body has a positive acceleration of 3 fps^2. It then proceeds with the velocity already acquired. Draw the time-velocity diagram with the final velocities for the first 10 sec. Plot time horizontal.

9–17. A bicycle and rider start from rest at the top of a hill. At the end of 10 sec the foot of the hill is reached with a speed of 20 fps. Find the acceleration (considered as uniform) during the descent.

9–18. If an automobile starts from rest with an acceleration of 8 fps^2, how fast will it be moving at the end of 10 sec? How far will it have moved?

9–19. If an automobile is traveling at 50 fps, and the brakes are applied with a negative acceleration of 8 fps^2, how many seconds will it take to stop? How far will the car travel in that time?

9–20. An automobile is traveling at 40 mph. (*a*) If the maximum braking effect causes a negative acceleration of 19 fps^2 what is the *braking distance*

(to bring car to a stop)? (*b*) If the driver's reaction time (to move his foot to the brake pedal) is $\frac{3}{4}$ sec, what is the *reaction distance*? (*c*) Under these conditions, what is the *stopping distance*?

9–21. A body is going 28 fps, when its velocity is increased 4 fps each second for 6 sec. Find the velocity at the end of that time and the distance traveled.

9–22. The velocity of a body changes from 400 fps to 180 fps in 5 sec. Find the acceleration and the distance traveled meantime.

9–23. An acceleration of 30 in feet and second units equals what acceleration in feet and minute units?

9–24. A railroad train starts from rest at a station and increases its speed uniformly at the rate of $\frac{1}{2}$ fps each second. How fast will it be going in miles per hour at the end of 20 sec? How far will it be from the station?

9–25. A body moving at the rate of 500 fps has its speed uniformly accelerated until at the end of 10 min its speed is 1000 fps. Find the acceleration in feet per second per minute.

9–26. A train running at rate of 30 mph is stopped by a sudden application of the brakes. What negative acceleration must be given in order that the train may be brought to rest (at a uniform rate) in 20 sec? In a distance of 300 ft?

9–27. In what time will the velocity of a body falling from rest become 100 fps if the acceleration due to gravity is 32 fps^2?

9–28. A ball is shot vertically upward with a velocity of 640 fps, and the acceleration of gravity is 32 fps^2. How long before the velocity of the body will be: (*a*) 512 fps upward? (*b*) Zero? (*c*) 96 fps downward? (*d*) 640 fps downward?

9–29. An automobile is moving uniformly at the rate of 3 fps. What uniform acceleration must be given it so it may go 130 ft in 10 sec?

9–30. An elevator starting from rest moves with an acceleration of 1 fps^2 for 10 sec. Find the final velocity, average velocity, and distance.

9–31. A moving body has its velocity changed uniformly from 1500 fps to 300 fps in 30 sec. Find the negative acceleration. How far will the body go meantime?

9–32. A toboggan sliding down a grade decreases its velocity from 20 fps to 4 fps. If this is done with a uniform negative acceleration of 2 fps^2, what time elapses during the change? How far does the toboggan go meanwhile?

9–33. Find the acceleration of a body which starting from rest acquires a velocity of 300 fps while moving through a distance of 1500 ft.

9–34. A body starting from rest increases its speed each second 5 fps. How far will it move during the fourth second?

9–35. A wheel making 5 rps has its velocity of rotation increased each second 30° per sec. How many revolutions per second will it be making at the end of 10 sec?

9–36. A street car has its speed diminished from 40 fps to 15 fps. If the brakes were applied with such force as to diminish the car's velocity 5 fps^2, how far did the car go in the meantime?

9–37. To what height and for what time will a body rise if projected vertically upward with a velocity of 100 fps?

9–38. With what initial velocity must a body start in order to rise vertically 700 ft before losing all its velocity?

9–39. What initial velocity must a body have if it is to move with a positive acceleration of 4 fps^2 for 10 sec, and pass over 500 ft?

9–40. A body going 150 fps has its speed decreased to 50 fps in going 240 ft. Find the negative acceleration.

9–41. A ball is thrown vertically downward from the top of a tower 550 ft high. In 5 sec it is 25 ft from the ground. What was its initial velocity and in what time will it reach the ground?

9–42. A ball is dropped from a certain point and $\frac{1}{4}$ sec afterwards another ball is dropped from the same point. How far apart will they be when the second ball has fallen 2 sec?

9–43. A balloon is rising at the rate of 20 fps when a sand bag is dropped from it. Ignoring air resistance, in what time and with what velocity will the bag reach the ground if the balloon was half a mile high when the bag was dropped?

9–44. A stone thrown vertically upward from the ground under an arch is seen to just touch the arch and return to the ground 4.5 sec after it started. How high is the arch?

9–45. Find the acceleration if a train starting with a velocity of 15 mph passes over 350 ft in 10 sec.

9–46. With what velocity must a body be projected vertically upward to attain a height of 6000 ft?

9–47. A wheel starting from rest is given an acceleration of 3.4 radians per sec^2. Through what distance will a point 2 ft from the axis of rotation move in 20 sec?

9–48. The velocity of a pair of rollers in a paper mill changes from 80 rpm to 120 rpm, in 30 sec. If the paper is passed between them without slippage, what length of paper will be passed in the 30 sec, assuming the diameter of the rolls as 20 in.?

9–49. Suppose a man is driving at 35 mph at night, when his lights pick up an obstruction 100 ft ahead. Suppose his reaction time is less than average, so allow $\frac{5}{8}$ sec. Suppose his braking effect gives a negative acceleration of 20 fps^2. Can he stop before reaching the obstruction?

64. Path of bodies projected horizontally or at an angle of elevation less than 90°. If a body is projected from a height in a horizontal direction, the velocity of projection and the acceleration of gravity are at right angles to each other and the body will move in a curve. The position of the body at any time, if we disregard air resistance, may be determined by remembering that the horizontal velocity will in no way affect the rate of vertical fall under

the influence of gravity, nor will the fact that the body is falling in any way affect the uniform horizontal velocity with which it started. This is really an illustration of the truth of "Newton's Second Law of Motion." Thus, if a body is projected horizontally from the top of a tower with a velocity of 400 fps, it will move in 3 sec a distance of $3 \times 400 = 1200$ ft out from the tower. In the same time it will, of course, have fallen with an acceleration of 32 fps^2.

Its average vertical velocity will be $\frac{1}{2}$ $(32 \times 3) = 48$ fps, and the vertical distance fallen $48 \times 3 = 144$ ft. Then at the end of the 3 sec, if we neglect modifications of its motion due to the air through which it has passed, it will be 144 ft below the top of the tower and 1200 ft out from it.

If the positions of a body under the conditions just described are calculated for several intervals of time and the results plotted, a curve, as in Fig. 147, will be obtained showing the position of the body at every instant, and the general shape of its path. This curve is a *parabola.*

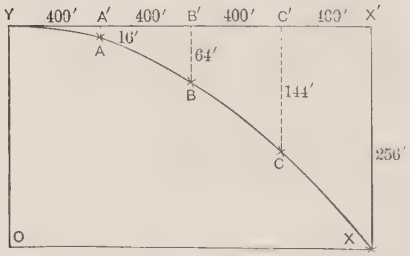


Fig. 147. Path of a body projected horizontally.

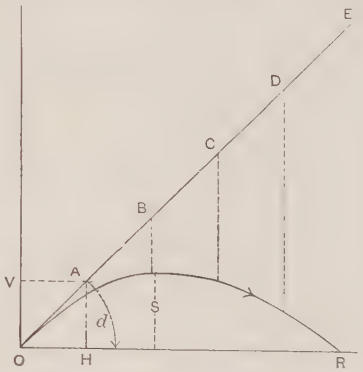


Fig. 148. Path of a body projected at an angle above the horizontal.

Suppose a body to be projected from O, Fig. 148, along the line OE, making an angle d with the horizontal. Let OA represent the velocity with which the body is projected from O. Then if gravity did not act, the body in 1 sec would be at A, in 2 sec at B, in 3 sec at C, in 4 sec at D, etc., where $OA = AB = BC$, etc. But since gravity also acts, the body will be *below* A at the end of 1 sec, a distance *equal to the distance it will fall in* 1 *sec*, i.e., it will be 16 ft vertically below A. Similarly, at the end of 2 sec it will

be 64 ft vertically below B, etc. In this way we may construct a curve giving the path of the body when we have given the amount and the angle with the horizontal of the original velocity.

The *greatest height*, S, and *range*, OR, of this curve may be computed as follows:

Resolve OA into its component horizontal and vertical velocities, H and V.

$$H = OA \cos d \quad \text{and} \quad V = OA \sin d$$

from which the numerical values of H and V may be calculated if OA is given. The horizontal and the vertical motions now may be

FIG. 149. Guns elevated to give greater range.

considered separately. The component V is the *initial* vertical velocity and H is the *uniform horizontal* velocity. Thus, $V/32 = t$, the time during which body will continue to rise. Therefore,

$$S = \left(\frac{V + 0}{2}\right) \times t$$

Since the body will require the same time to fall as was required to rise, the whole time of flight will be $2\,t$.

Therefore the "horizontal range," $OR = H \times 2\,t$. No allowance is here made for air resistance, wind, etc. These effects are

variable with the shape of the projected body, initial velocity, etc., and special corrections must be applied for any actual case of projection, as for example, in the firing of long-range guns.

PROBLEMS

9–50. A stone is thrown horizontally out from the top of a cliff 248 ft high with a velocity of 70 fps. Draw a diagram to scale showing the starting point, point at which it reaches the ground, and four intermediate positions.

9–51. When and where will a ball strike the ground if it is thrown horizontally out from the top of a tower 200 ft high with a velocity of 20 fps?

9–52. A body has simultaneously a uniform velocity due north of 40 fps, and an acceleration due east of 20 fps^2. Find its position at the end of the first, second, third, fourth, and fifth seconds.

9–53. Plot the path of the body in Problem 3, and from the curve take the position of the body at the end of 3.6 sec.

9–54. A projectile is fired at an angle of 30° to the horizontal with a velocity of 144 fps. Draw a diagram of its path showing its position at the end of each of the first 4 sec, no correction being made for air resistance. In what time will it reach the ground?

9–55. Find the greatest height reached and the horizontal range of a projectile, neglecting air resistance, the initial velocity of which is 950 fps at an angle of 60° to the horizontal.

65. Simple harmonic motion. A third type of motion, as distinguished from uniform motion and uniformly accelerated motion, is illustrated in the periodic swinging back and forth between the two extreme positions of a pendulum-bob, of the free end of a flexible rod when clamped at one end and then pulled aside and suddenly released, of a weight suspended from a spring, etc. Motion of this sort is known as **simple harmonic motion.** All vibrations due to the elastic properties of bodies are simple harmonic. The to-and-fro motion of the cross-head of a steam-engine is approximately simple harmonic.

The **characteristics** of simple harmonic motion are:

1. It is a *periodic* oscillation back and forth between two extreme positions.

2. The *velocity* is maximum at the center of the path, and zero at each end.

3. The *acceleration* is toward the center of the path.

4. The acceleration is zero at the center and maximum at each end; its amount is *proportional to the distance* from the center (displacement).

5. The *period* (time required for a complete oscillation, round trip) varies directly as the square root of the displacement and inversely as the square root of the acceleration.

Expressed in symbols the equation for simple harmonic action is

$$T = 2\pi\sqrt{\frac{x}{a}} \tag{15}$$

in which T represents the period, x the displacement, and a the acceleration.

The truth of the foregoing five statements can be illustrated by the **reference circle** of a simple harmonic motion. Let a point P' move with a *uniform velocity* v in a circular path of radius OA as in Fig. 150 and let P be the projection of P' on the diameter AB, that is, imagine a perpendicular dropped from each successive position of point P' to the diameter. Then, as P' moves around the circle with *uniform* motion, point P moves periodically back and forth between the two extreme positions A and B with *simple harmonic* motion. The time required for P to move from A to B, and then back again from B to A, is called the **period.** It is evidently the time required for P' to travel once around its circular path. The number of periods per unit time is called the **frequency.** The radius OA is called the **amplitude.** The distance from O to P at any instant is called the **displacement.**

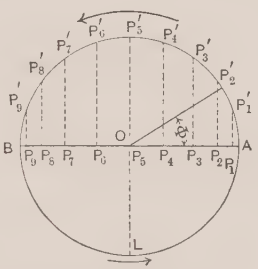


FIG. 150. Reference circle for simple harmonic motion.

Inspection of the figure shows also that while P' *moves with uniform velocity*, passing over the equal arcs, AP'_1, $P'_1P'_2$, etc., in successive equal intervals of time, point P travels in the corresponding time intervals over distances AP_1, P_1P_2, etc., which *are unequal* and which *increase as* P *moves toward the center* O, and *decrease as* P *moves away* from the center. The velocity of P along the diameter at its extreme positions A and B, when P' is moving for an instant at right angles to the diameter, is zero. At P'_5 and the corresponding position on the other semicircumference, P' is moving for an instant parallel to the diameter AB, hence at these points, P, which is passing through O, moves with a maximum velocity which is equal to the velocity of P', or v. Starting from an extreme position as A, therefore, with a velocity

of zero, P moves with increasing velocity along its path to O where its velocity has a maximum value v. From O to B, the velocity of P decreases, becoming zero when B is reached. Similar changes in velocity take place as P moves back over its path from B to A.

66. Phase of a simple harmonic motion. Since both the determining factors of a motion, velocity and acceleration, are dependent in simple harmonic motion upon the position of the moving body in its path, it is convenient to have some easy way of fixing the stage arrived at in the oscillation. Thus while point P in Fig. 150 is the *body actually moving* with a harmonic motion, it is convenient to project P back upon an imaginary circle and fix its position, and therefore all conditions of the harmonic motion, by locating point P' on the reference circle. The position of the imaginary radius vector OP', assumed to start from the reference line OA, Fig. 150, and to rotate in the direction of the arrows, then shows clearly the stage which has been reached in the simple harmonic motion. The angle ϕ between this radius vector and OA, is known as the **phase** of the simple harmonic motion.

67. The sine curve. If the point P, moving as in Fig. 150, were able to make a tracing upon a surface held under the circle,

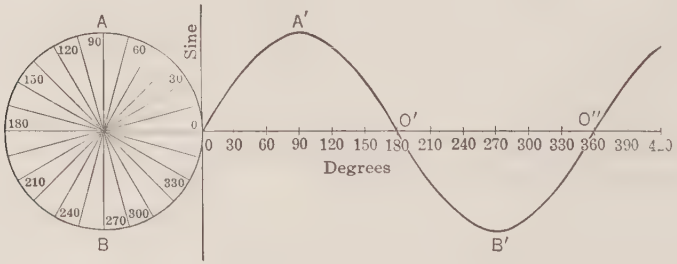


Fig. 151. The sine curve.

and if this surface were moved uniformly in a direction at right angles to AB, the result would be a curve, as $OA'O'B'O''$ in Fig. 151, in which the diameter AB has been rotated into a vertical position, and the start of the curve is with P at the center of its path. This curve, which is characteristic of simple harmonic motion, is called a **sine curve** since the ordinates of points on the curve are proportional to the sines of the corresponding angles of phase.

If we imagine the abscissas of this curve to represent time, or some corresponding quantity, the sine curve represents clearly the successive stages in the simple harmonic motion. Curve $OA'O'B'O''$ represents a complete cycle of changes. Sine curves are commonly used in representing periodic changes in an alternating current, in alternating-current voltages, etc.

68. The pendulum. A simple *pendulum* consists of a ball suspended by a thread so light and flexible that its mass and stiffness may be disregarded. Such a pendulum furnishes a practical illustration of simple harmonic motion, for, if set vibrating, the ball will pass periodically back and forth over its path, with a maximum velocity at the middle point of its swing, and zero velocity at the ends, where the direction of motion reverses.

Experiment and theory both show that the period of a pendulum (i.e., the time for a complete oscillation) is independent of the mass of the pendulum-bob and practically also of the length of arc through which it swings, and dependent only upon the length of the pendulum and the value of gravity. The law of the pendulum is expressed by the equation

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{16}$$

where T is the period, L the length of the pendulum, and g the acceleration due to gravity at the locality. The equation shows that, other values being constant, **the period of a pendulum varies as the square root of the length. Also that the period varies inversely as the square root of the acceleration of gravity at the locality.**

The pendulum, in some form, furnishes one of the common and most accurate methods of determining the value of g, the acceleration due to gravity of a freely falling body. If a simple pendulum of known length is oscillated and its period found, the values for the length and period may be substituted in the foregoing equation and the expression solved for g. Various special forms of pendulum have been devised to permit accurate determinations of period and length, descriptions of which can be found in textbooks on physics.

Other methods for determining the acceleration of gravity include some form of the Atwood's machine, on the principle of the apparatus suggested by Fig. 156; some form of rolling body on an inclined plane apparatus; or some form of free fall apparatus, this

latter being equipped, in its modern form, with an electrical timing device and an electrical means of making a permanent record on a paper ribbon along which the body falls.

69. Cam and linkage motions. A **cam** is usually a plate or cylinder which has a motion of rotation and which causes motion of a follower (usually a sliding member) by means of its edge or a groove cut in its surface. For each complete revolution of the cam the follower makes one complete trip (out and return) over its path. Its position at any instant of its movement is definitely fixed by the shape of the cam. This displacement ratio, rather than speed ratio, is the important feature of the study of cam motion. The three most common forms of motion desired in the follower are (*a*) uniform, (*b*) harmonic, and (*c*) uniformly accelerated and retarded.

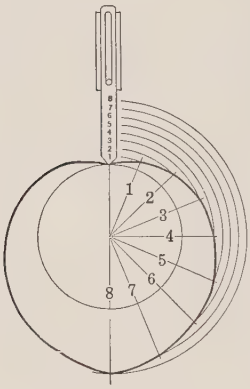


FIG. 152. Model cam and follower. A "heart" cam causes uniform motion.

The study of the cam mechanism is most easily done graphically, since it is the path and amount of motion (displacement) which is desired. An idea of the method can be gained by considering, for example, the shape of a plate cam desired to lift a follower 8 in. in a straight line and then lower it the same amount at uniform speed for each revolution. (Fig. 152.) This cam will be found to be of the "heart" type, with the shaft center 8 in. nearer one edge of the cam than the opposite, so that in a half revolution the point of the follower will have been lifted the required amount. If the radius of the top of the cam with the follower in bottom position is, say, 8 in., then when the cam has turned one-eighth of a half revolution the top radius will measure 9 in., at one-quarter 10 in., at three-eighths 11 in., etc., until at the full half turn the radius has become 16 in. and the follower is in top position. The cam is symmetrical so that the radii decrease uniformly in like manner throughout the second half revolution.

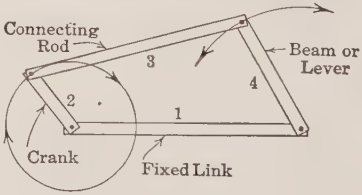


FIG. 153. Model of four link "beam and crank" chain.

Linkage is a form of mechanism in which four or more parts are joined together in "a chain." The simplest chain is made of four straight bars fastened together at their ends by cylindrical pins. See Fig. 153. One of these links is fixed and the other three movable in parallel planes. With links of different lengths, and with different links being fixed varying forms of motion will result.

The *beam and crank* mechanism is the most common four-link chain. The fixed link is the frame of the machine, the second link

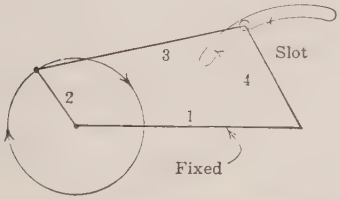


FIG. 154. Center line diagram of preceding figure, with slotted "link."

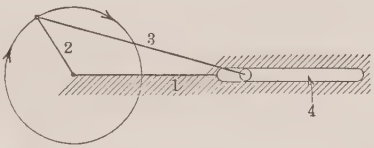


FIG. 155. "Sliding-block" chain.

is the crank, rotating, usually as a wheel or gear, the third is a connecting rod, and the fourth is the beam or lever oscillating back and forth. By making a change in the joint between connecting link and beam from a pin to a slotted arrangement, and further by making the slotted piece a straight line, giving the effect of an "infinite length" to the beam link, the much used "engine" or "sliding block" chain is developed. See Figs. 154 and 155.

PROBLEMS

9–56. If the period of a certain pendulum is 1.2 sec, what is the period of one three times as long?

9–57. If a pendulum of length 30 in. has a period .885 sec, what is the period of one 20 in. long?

9–58. A pendulum whose length is 6 makes 40 vibrations per min. How many will one whose length is 2 make in the same time?

9–59. A certain pendulum is found to make one vibration in .8 sec at a place where the value of g is 31.8. What will be its period at a place where g is 32?

9–60. Calculate the length of a pendulum whose period is 2 sec.

9–61. The period of a pendulum 39.05 in. long is 2.01 sec. From these figures find the value of g.

9–62. The rope carrying the car of a mine shaft is 900 ft long. What will be the period of swing as a pendulum?

9–63. Draw the outline of a cam with its least radial dimension 3 in. to lift a follower 6 in. at uniform speed and return at same speed. (Suggestion: draw the 3-in. circle to one-half or one-quarter scale, and divide into 30° angles, six divisions in a half revolution. Refer to Fig. 152.)

9–64. A simple chain consists of four links: A is 9 in. long, B 2.5 in., C 10 in., D 4.5 in. Consider A as fixed and B as a rotating member. Make a plot to scale (using single lines, as Fig. 154) showing the extreme left and right positions of link D and measure the angular displacement of D with a protractor.

9–65. If the crank B in Problem 9–64 is the driver, at 120 rpm counterclockwise, find the time interval of each stroke of D (*a*) to the left, (*b*) to the right. (The crank will sweep through a different angle for the left stroke of D than for the right stroke.)

9–66. Plot the sine curve for a reference circle of 2-in. diameter. (Suggestion: Use a horizontal scale of $\frac{1}{4}$ in. = 15° for angular motion, and a vertical scale of 1 in. = 1 in. for displacement.)

9–67. Plot a curve for the displacement of the cam follower in Problem 9–63. (Cam angle horizontal, displacement vertical.)

9–68. Substitute the specification of simple harmonic motion for the follower in Problem 9–63 and draw the displacement graph and outline of cam.

SUMMARY

Acceleration is the rate of change in velocity per unit time. Acceleration = change in velocity ÷ time. It is expressed in such units as feet per second per second, which is written fps per sec, ft per sec^2, or fps^2. An increase in velocity is a positive acceleration; a decrease in velocity is a negative acceleration, or a retardation.

Falling bodies are uniformly accelerated under the influence of gravity. The value of the acceleration of gravity is approximately 32.2 ft per sec^2 at sea level. For most problem work an even 32 ft per sec^2 is sufficiently exact.

Projectiles act under the influence of gravity as well as the force of projection.

Simple harmonic motion is a type of motion, neither uniform nor uniformly accelerated. The swinging, or oscillation, of a pendulum is an illustration of this type. Its characteristics, and the use of the reference circle, are treated in Art. 66. The sine curve is a graphic representation of simple harmonic motion.

A **cam** is a machine mechanism which transfers circular motion to linear motion in the cam follower.

The **simple four-link chain** is another type of machine motion, in which motion of rotation in one link causes, or is caused by, oscillating or sliding motion in another link.

Many problems are solved best by an application of the fundamental principles of accelerated motion rather than by formula.

The formula is, however, an aid to the memory and its use may result in saving time.

The general fundamental formulas are re-stated below, in symbols for linear units. To derive the special equations for motion from rest, let $v_0 = 0$; for falling bodies, let $a = g$ and $s = h$; for angular units, substitute ω for v ($\omega = v/r$), θ for s, and α for a.

$$a = \frac{v_c}{t} \quad \text{and} \quad v_c = at \tag{9}$$

$$v_f = v_0 + v_c \quad \text{also} \quad = v_0 + at \tag{11}$$

$$v_a = \tfrac{1}{2}(v_0 + v_f) \quad \text{also} \quad = v_0 + \tfrac{1}{2}at \tag{12}$$

$$s = v_a t \quad \text{also} \quad = v_0 t + \tfrac{1}{2}at^2 \tag{13}$$

One special equation also is included because of its frequent use in finding the relations between displacement, velocity, and acceleration, for motion from rest, without necessarily computing the time.

$$s = \frac{v^2}{2a} \quad \text{and} \quad v = \sqrt{2as} \tag{14R}$$

a = linear acceleration in feet per second per second.
g = acceleration of gravity = 32 fps^2.
r = radius in feet.
s = linear displacement in feet.
v = linear velocity in feet per second.

v_0 = initial velocity.
v_c = change in velocity.
v_f = final velocity.
v_a = average velocity.

α (alpha) = angular acceleration in radians per second per second.
θ (theta) = angular displacement in radians (= revolutions $\times 2\pi$).
ω (omega) = angular velocity in radians per second.

CHAPTER X

FORCES PRODUCING MOTION

Chapter X continues the study of *motion*, also taking into consideration the *forces* causing the motion.

Many students, today, are probably more familiar with the forces and motion in the quick starting and stopping automobile than with the heavier railroad train. Their attention has not been attracted so often to the need for putting on the brakes a half mile before they expect to stop, although they probably have observed that even an automobile requires more braking effort when loaded than when light, or when traveling at a high rate of speed than when moving slowly or only moderately fast. The need for banking the track on a curve is less apparent when riding in a 3000-lb automobile than in a 300-ton train, although the effect of centrifugal force is felt in the automobile, particularly if the radius of the curve is short. Riding in one of two trains on parallel tracks around a curve the passenger finds himself much higher or lower than the windows in the coach beside him on the adjacent track.

In this chapter the material to be studied includes the important relationship between force and the mass of a body as it affects the resulting motion of the body; also the inertia reaction, that property by which bodies tend to resist a change from rest to motion, or vice versa. The meaning of the moment of inertia and the radius of gyration will be considered; and the method of computing centrifugal force.

70. Unbalanced forces. It has been shown (Art. 20 and Art. 21) that a body will continue at rest or moving with uniform velocity in a straight path or rotating with uniform velocity about a fixed axis when, for every force that would produce displacement in any given direction, there is an exactly equal force in the opposite direction; and for every torque that would cause the body to rotate in one direction about a given axis, there is an equal torque about the same axis in the opposite direction. These conditions of balanced force and balanced torque furnish the necessary equations for the solution of problems having to do with systems of forces in equilibrium.

There are many important cases in mechanics, however, where *the system of forces acting is not in equilibrium.* Familiar examples are all cases in which a body is being started or stopped in either motion of translation or rotation or where it is being made to move in a *different direction.* In all such cases there is an unbalanced force which is causing the body to change its motion in the direction of the unbalanced force or an unbalanced torque which is causing it to rotate faster or slower. The equations for forces in equilibrium do not apply, and it becomes necessary to be able to compute the unbalanced force or the unbalanced torque which is required to produce the given change in the motion of the body under consideration.

71. Factors affecting acceleration. It is a matter of common observation that the greater the tractive effort the engine will give, the quicker an automobile will " pick-up " a given speed, and the greater the braking effect that can be brought into use, the quicker a moving automobile can be stopped. It is equally a matter of common experience that it is more difficult (i.e., requires more force) to impart a given motion in a stated time to a loaded car than to an empty one or to stop a heavy body in the same time than to stop a lighter one moving with the same velocity. Reference to Newton's Second Law furnishes a means of determining the relation between the acceleration of a body and the force applied, also between the acceleration and the weight or mass. If a force F_1 is applied to a body causing it to move with an acceleration a_1, and then another force F_2 is applied causing an acceleration a_2, according to Newton's law the two accelerations will be in the same proportion as the respective forces. That is

$$\frac{F_1}{F_2} = \frac{a_1}{a_2} \tag{17}$$

Suppose the second force is the gravity pull on the body, i.e., suppose the body is allowed to fall from rest under its own weight. Then the acceleration will be that of a falling body, 32 fps^2. The equation then becomes:

$$\frac{F}{W} = \frac{a}{g} \quad \text{or} \quad \frac{F}{a} = \frac{W}{g} \tag{18}$$

A consideration of this proportion shows that the force has the

same ratio to the acceleration caused as has the weight to the acceleration of gravity. It also shows that, if the weight of the body were changed, the force would have to be changed in the same proportion in order to keep the acceleration constant. This proportion makes it possible to determine the effect of a force in moving a body.

Example.

What acceleration will be given to an automobile weighing 2000 lb by a push of 50 lb (neglecting the amount required to overcome friction)?

Solution.

From (18) $\frac{F}{W} = \frac{a}{g}$ $\frac{50}{2000} = \frac{a}{32}$ $a = \frac{50 \times 32}{2000} = 0.8 \text{ fps}^2$ *Ans.*

The laboratory apparatus illustrated in Fig. 156 furnishes a means of arriving at a quantitative expression of the qualitative proportion just stated. W is a light wheel over which two equal masses A and B are hung by a light flexible cord. R is an extra mass on A. To secure accurate results the apparatus must be frictionless and the masses of wheel and cord so small that they may be neglected.

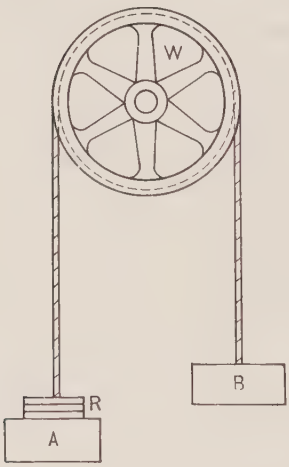


Fig. 156.

Suppose A and B are each 1.5 lb and R is 0.2 lb, making a total mass of 3.2 lb weight. When released A will move down and B will be drawn up with uniformly accelerated motion. By measuring the distance A moves in a known length of time, the acceleration of the apparatus can be calculated.

Under the conditions assumed, A would move downward 4 ft in 2 sec time; from which the acceleration can be computed to be 2 fps^2. Thus $s = \frac{1}{2} at^2$ equation (13R); from which $a = 2\,s/t^2 = (2 \times 4)/2^2 = 2$ fps^2.

If now R is changed to 0.4 lb, and A and B are reduced to 1.4 lb each, thus leaving the total mass to be moved the same 3.2 lb but the unbalanced, or moving, force of 0.4 lb twice as much as before, the distance moved in 2 sec would be found to be 8 ft. The

acceleration $= 2\ s/t^2 = 2 \times 8/4 = 4$ fps^2, which is twice as much as before. This is in accordance with the proportion previously stated.

Suppose the experiment is extended by doubling the mass and again using first 0.2 and then 0.4 lb for the rider. When A and B are each 3.1 lb and R is 0.2 lb a total of 6.4 lb, the distance would be 2 ft in 2 sec, from which $a = (2 \times 2)/4 = 1$ fps^2. When A and B are each 3.0 lb and R is 0.4 lb, the distance moved in 2 sec should measure 4 ft, from which $a = (2 \times 4)/4 = 2$ fps^2. The results are tabulated for comparison.

	Moving Force	Total Mass	Acceleration
Case 1	Gravity pull on 0.2 lb	3.2 lb	2 fps^2
Case 2	" " " 0.4 lb	3.2 lb	4 fps^2
Case 3	" " " 0.2 lb	6.4 lb	1 fps^2
Case 4	" " " 0.4 lb	6.4 lb	2 fps^2

To summarize the foregoing considerations, it may be concluded that (1) **with the same mass the moving force is proportional to the acceleration,** i.e., $F \propto a$; and (2) **with the same acceleration the force is proportional to the mass,** i.e., $F \propto m$.

$$F \propto ma \tag{19}$$

72. Mass and weight. In Art. 71 the term " mass " was used to measure the quantity of matter moved, and the term " gravity pull " on a given mass to measure the moving force. The gravity action or force with which the earth pulls on a body is called the weight of the body.

The expressions mass and weight, therefore, although referring to the same body, do not represent the same idea. *Mass*, by whatever conception it is measured and in whatever units it is expressed, *always signifies quantity of matter*. The mass of a given body does not change as the body is carried from one locality to another. The inertia of a body (Art. 7) is dependent upon its mass. More force is required to impart a given acceleration to a larger mass because greater inertia is opposed to the change of motion.

On the other hand, the *weight* of a body, that is, *the force with which the earth pulls upon it, depends upon its position on the earth's*

surface. The change in the earth's pull with the locality is not great and may, except for very exact computation, be disregarded. In general, it is less at the equator and greater as the poles are approached.

73. The gravitational units of force and mass. Thus far the relation between force, mass, and acceleration has been expressed as a *proportion* only (Formula 19). The proper selection of units makes it possible to express it as an *equation*, and to solve for definite amounts.

In common usage, the word *pound* has two meanings. It is used as a unit of quantity of material, as a pound of sugar, iron, etc., and so used is a unit of mass. It is also used as a unit of push or pull and so used as a unit of force. English-speaking engineers find it most convenient to use a pound or some multiple of a pound as a unit of force.

The unit of mass is taken as a quantity having the same proportion to 1 lb, as the acceleration of gravity g has to one. This unit is called a *slug* and equals a mass of *approximately* 32-lb weight. It is so chosen because it is the mass which would be accelerated at the rate of 1 fps^2 by the action of a force of 1 lb. With these units $m = W/g$ and:

$$F = ma \quad \text{or} \quad F = \frac{W}{g}a \tag{20}$$

which is in agreement with equation (18).

Example 1.

How much tractive force (friction neglected) would be necessary to give the locomotive in Fig. 157 an acceleration of 1 fps^2?

Solution. Here

$$F = \frac{700{,}000}{32} \times 1 = 21{,}900 \text{ lb} \quad \textit{Ans.}$$

Example 2.

A car weighing 4000 lb is moving at the rate of 30 fps. What average force will stop it in a distance of 150 ft?

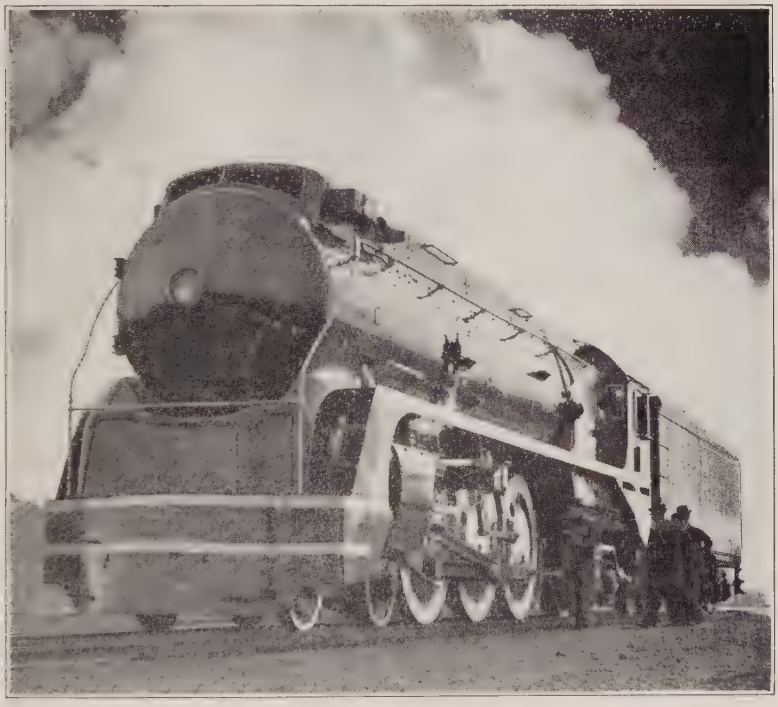
Solution. As the final velocity is zero, the average velocity is 15 fps and the time $\frac{150}{15} = 10$ sec. The acceleration is therefore $\frac{30}{10} = 3$ fps^2.

Hence $$F = \frac{4000}{32} \times 3 = 375 \text{ lb} \quad \textit{Ans.}$$

The metric or cgs unit of force is the dyne, which is that amount of force that will give a mass of one gram an acceleration of 1 cm per sec^2. In this

case $F = ma$ as before in which each unit is 1, that is 1 dyne = 1 gram $\times$ 1 cm per sec^2.

Another unit of force is the poundal and is that force which will give a mass of 1 lb an acceleration of 1 ft per sec^2.



Courtesy The New York, New Haven and Hartford Railway Company.

FIG. 157. Modern streamlined steam locomotive and tender. Length 97 ft.; weight 700,000 lb.; tractive force 44,000 lb.

74. Motion of a body on a smooth incline. The acceleration of a body falling freely from rest is the direct result of the gravity pull, i.e., its weight. The acceleration of a body sliding down an inclined plane is due to the component of its weight which acts along, parallel to, the plane. If the plane could be absolutely smooth, there would be no friction between the plane and the body to prevent sliding and none of the *gravity component* of the weight would be used up to overcome friction.

In Fig. 158 OA represents the total pull of gravity W on a body O. Then OB and OC represent the components of W parallel and

perpendicular, respectively, to the face of the incline. By geometrical construction $\angle OAB = \angle RST = \angle\delta$, between the plane and the horizontal. In triangle AOB, $\sin\delta = OB/OA$; in triangle RST, $\sin\delta = RT/SR$. Therefore

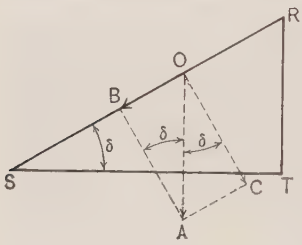


Fig. 158.

$$OB = OA \sin\delta = W \sin\delta = W \times \frac{RT}{SR}$$

If friction is neglected, the component OC does not affect the motion, and the moving force is OB, tending to give motion down the plane. Since the full force of gravity W will give an acceleration g, the force $OB = W \times RT/SR$ will give an acceleration

$$a = g \times \frac{RT}{SR} = g \sin\delta \qquad (21)$$

OC is called the *normal* component (N), and OB the *gravity* component (G).

75. Acceleration diagrams and the solution of problems. When a body is made to move with accelerated motion in any direction, the force F_1 in the direction of motion must always be

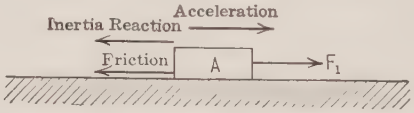


Fig. 159. Acceleration diagram for a body moving with increasing speed along a rough, horizontal surface.

greater than the force, F_2, opposing motion. The difference, $F_1 - F_2$, represents the *unbalanced force causing acceleration.* Opposed to this unbalanced force is an *equal reaction due to inertia*, or $F_1 - F_2$ = inertia reaction = $\frac{\text{weight of body or bodies moved}}{g}$ $\times$ acceleration. If we include this inertia reaction as an *equivalent force opposing change of motion*, we may write as the general equation for accelerated motion:

Sum (all actions in the direction of the change of velocity) = Sum (all reactions in the opposite direction)

Thus suppose a body A is made to slide with increasing speed over a rough horizontal surface, Fig. 159, by a pull F_1.

Acceleration occurs in the direction of F_1; the reactions are due to friction and inertia.* Therefore, by the general equation,

$$F_1 = \text{friction} + \text{inertia reaction} \left(\text{i.e.,} \frac{\text{wt. } A}{g} \times \text{acc.} \right)$$

If a body A, already in motion, is being brought to rest by friction, our " acceleration diagram " will be as in Fig. 160. Fric-

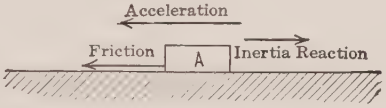


FIG. 160. Acceleration diagram for a moving body being stopped by friction.

tion is decreasing the velocity (i.e., producing a negative acceleration), and the inertia of A is tending to continue its motion. Therefore,

$$\text{Friction} = \text{inertia reaction} = \frac{\text{wt. } A}{g} \times \text{acc.}$$

In general, in the solution of problems, proceed as follows:

1. Determine the direction of the change in velocity.

2. Construct an " acceleration diagram," showing all actions in the direction of the change in velocity, and all reactions in the opposite directions. Always include among the latter, one due to inertia.

3. Substitute the values for these actions and reactions in the general equation and solve.

Example 1.

Weights of 90 and 80 lb hang by a cord over a pulley as in Fig. 161. Find the acceleration of the system, if friction and the mass of the cord and wheel may be neglected.

* The student must not confuse what is here called the " acceleration diagram " with the force diagram of a free body in equilibrium. The moving body is not in equilibrium. In reality, the force F_1 Fig. 159, which causes A to move, comes from some second body B. *The inertia reaction of A is therefore really* exerted *on B and not on body A.* Fig. 159 is therefore not a force diagram for body A. The student will, however, find such " acceleration diagrams " helpful in more complicated cases of accelerated motion, because they help to create a clear idea of the true conditions, and tend to prevent the omission of any forces or reactions affecting the motion of the given body.

Solution. Here the total weight moved, W, is 80 + 90 = 170 lb. The force in the direction of the acceleration is 90 lb. The opposing reactions on the moving system are the gravity pull of 80 lb on B and the inertia reaction of the system. Therefore,

$$90 = 80 + \tfrac{170}{32.2} \times a$$

from which,

$$a = 1.88 \text{ fps}^2 \quad Ans.$$

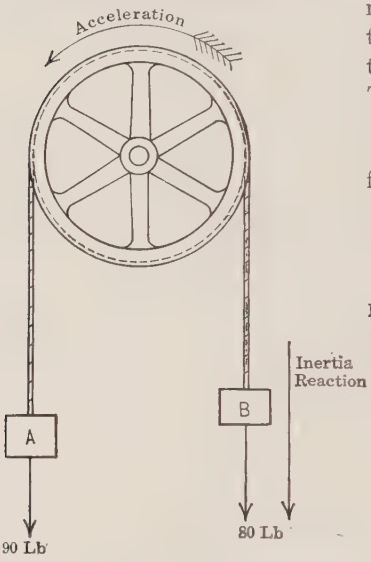


Fig. 161. Acceleration diagram for unequal weights hung over a freely moving pulley.

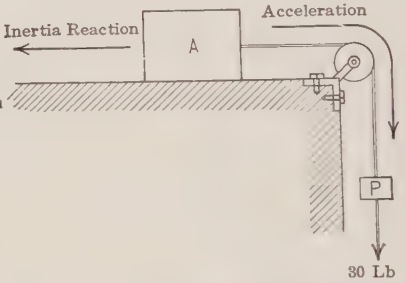


Fig. 162. Acceleration diagram for a body moved along a horizontal surface by a falling weight.

Example 2.

In Fig. 162, A = 500 lb and P = 30 lb. If there is no friction, find the acceleration and the time required for the system to move 6 ft.

Solution. Here the total weight W to be moved = 500 + 30 = 530 lb. The gravity pull on A is borne by the surface and the gravity pull on P alone causes motion. Therefore

$$30 = \tfrac{530}{32.2} a$$

$$a = 1.81 \text{ fps}^2 \quad Ans.$$

The average velocity $= \frac{1}{2} at = \frac{1.81\, t}{2}$, therefore

$$6 = \frac{1.81\, t}{2} \times t$$

or

$$t^2 = \frac{12}{1.81}$$

$$t = 2.58 \text{ sec} \quad Ans.$$

Example 3.

A car weighing 1800 lb is drawn vertically up a mine shaft. In starting the car, the acceleration is found to be 20 fps^2. Find the tension in the cable while starting.

Solution. The "acceleration diagram" for the car is here as shown in Fig. 163.

The force causing motion is the pull T in the cable; the opposing reactions are the weight W of the car and the inertia reaction, which is $\frac{1800}{32} \times 20$.

(a) Therefore

$$T = 1800 + \frac{1800}{32} \times 20$$

From which,

$$T = 2925 \text{ lb} \quad Ans.$$

(b) Another solution may be given, thus: W = 1800 lb; and a = 20 fps^2. Let T_1 = pull of the cable to give the car an acceleration of 20 fps^2; then

$$T_1 = \frac{1800}{32} \times 20$$

$$T_1 = 1125 \text{ lb}$$

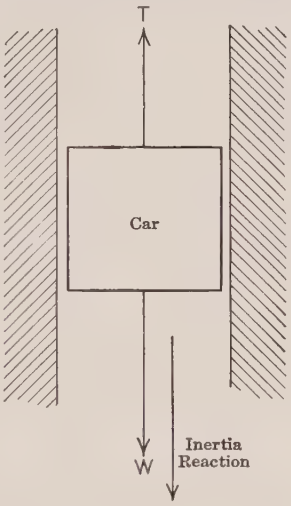


Fig. 163. Acceleration diagram for starting a car up a shaft.

The tension T in the cable is then the 1800 lb necessary to hold the car up plus 1125 lb required to give the upward acceleration, or 2925 lb. *Ans.*

Example 4.

In Fig. 164 suppose W is 90 lb, P 40 lb, the friction between W and the plane 3 lb, the friction of the pulley 2 lb. Find the direction and

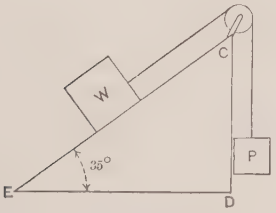


Fig. 164. A body moving on an incline.

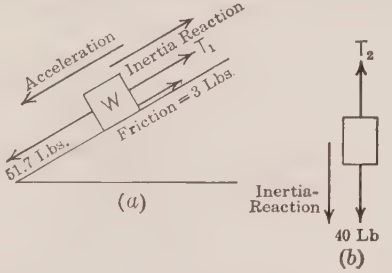


Fig. 165. Acceleration diagrams for the preceding figure.

the amount of the acceleration, and the tension in the cord on each side of the pulley.

Solution. The force tending to move the system so that W will move *down* the plane is 90 sin 35° = 51.7 lb. The forces opposing this motion are P, 40 lb, and friction, 5 lb. The force down the plane is greater and, therefore, the motion will be down the plane. The total weight to be accelerated is 90 + 40, or 130 lb, and the inertia reaction of the system is $\frac{130}{32} \times a$.

Therefore, we may write the equation

$$51.7 = 40 + 5 + \frac{130}{32} \times a$$

From which $a = 1.65$ fps²

To find the tension T_1 in the cord between W and the pulley, consider W alone. The acceleration diagram is as in Fig. 165(a).

Therefore, $51.7 = T_1 + 3 + \frac{90}{32} \times 1.65$

or $T_1 = 44$ lb *Ans.*

Then, from the pulley considered alone, we have, neglecting the mass of the pulley,

$T_1 = T_2 + 2$

$T_2 = 42$ lb, the tension on the other side

Or, T_2 may be found from the equation for P alone [Fig. 165(b)]

$T_2 = 40 + \frac{40}{32} \times 1.65$

$T_2 = 42$ *Ans.*

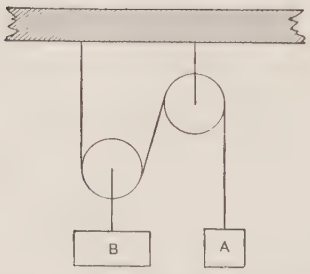


Fig. 166. A system of weights and pulleys.

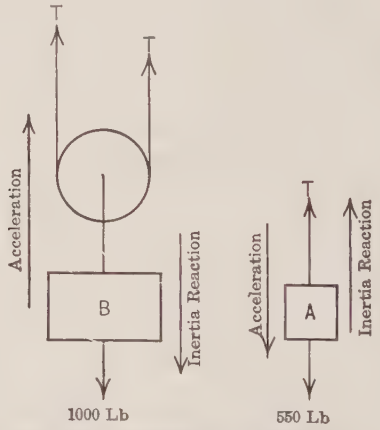


Fig. 167. Acceleration diagrams for the preceding figure.

Example 5.

A weight B of 1000 lb is supported by a combination of pulleys as in Fig. 166. If A is 550 lb, with what acceleration will the com-

bination move and what will be tension in the rope if friction and the weight of pulleys may be neglected?

Solution. It will be seen from the figure that the motion of A is twice that of B. Therefore the acceleration of A will be twice the acceleration of B.

Let a = the acceleration of B. Then $2\,a$ = the acceleration of A. Let T = the tension of the rope; since there is no friction the tension everywhere in the rope is the same. The diagrams for A and B are as in Fig. 167.

Therefore we may write the equations:

$$T + \tfrac{550}{32} \times 2\,a = 550 \qquad (1)$$

$$2\,T = 1000 + \tfrac{1000}{32} \times a \qquad (2)$$

These simultaneous equations may be solved for T and a in the usual way, giving $T = 516$ lb and $a = 1$ fps^2. *Ans.*

PROBLEMS

10–1. A constant force acting upon a mass of 15 lb for 4 sec gives it a velocity of 20 fps. Find the force.

10–2. A man pushes steadily against a car on a level track for 1 min and thereby produces a velocity of 12 fps. A horse in the same time can produce a velocity of 20.5 fps. Compare the force exerted by the man with that exerted by the horse.

10–3. Force A can produce a velocity of 100 in 5 sec, on a certain mass. Force B can produce on the mass a velocity of 50 in 25 sec. How many times as large as B is the force A?

10–4. A body at rest weighing 4 lb is struck and at the end of the blow has a velocity of 10 fps. Assuming the time the blow lasts to be $\frac{1}{100}$ of a sec, find the average force of the blow.

10–5. A car slides down a smooth incline having a slope of 1 ft in each 5 ft length of track. Find: (*a*) The acceleration; (*b*) the velocity acquired in 8 sec; (*c*) the distance traversed in 5 sec.

10–6. Sand will stop a certain projectile fired into it in $\frac{1}{10}$ of a sec. Loam will stop the same body going twice as fast in $\frac{1}{8}$ of a sec. Compare the resistances to penetration offered by sand and loam.

10–7. What velocity would a train acquire in half a minute on a 3 per cent grade if there were no friction and what distance would it travel in that time?

10–8. A body weighing 90 lb is moving at the rate of 80 fps. What force expressed in pounds will stop it in 2 sec? What force in $\frac{1}{10}$ of a sec?

10–9. In an apparatus such as shown in Fig. 161, A and B weigh 10 lb and 9 lb, respectively, and hang by a flexible string over a frictionless pulley. Find: (*a*) The acceleration of the system; (*b*) the distance traversed in 2 sec.

10–10. In an Atwood's machine, similar to Fig. 156, the weights carried by the thread are $6\frac{1}{2}$ oz each. The friction is equivalent to a weight of $\frac{1}{2}$ oz. When the "rider," which weighs 1 oz, is in position, what velocity will be gained by the system each second?

10–11. A train weighing 1000 tons is going 30 mph when the brakes are applied. The brakes can exert a force tending to stop the train of 10 tons; how long will it be before the train will stop if the track is level and we neglect all other friction? How far will the train go before stopping?

10–12. Answer Problem 10–11, if train is going up an incline of 1 in 20.

10–13. In an apparatus such as is shown in Fig. 162, A weighs 100 lb, P, 5 lb. If there is no friction, how fast will A be moving after going 10 ft along the table? How fast if friction is equivalent to a retarding force of 3 lb?

10–14. A man weighing 175 lb stands on the floor of an elevator which is descending with a uniform acceleration of 1 fps^2. What will be the pressure of his feet on the floor? What will be the pressure when the elevator is *ascending* with the same acceleration?

10–15. A rope winding on a drum hauls a car vertically up the shaft of a mine with an acceleration of 2 fps^2. If the car weighs $1\frac{1}{2}$ tons, what is the tension in the rope?

10–16. A railroad train is moving on a level track at the rate of 30 mph. The train weighs 400 tons. If the steam is shut off, what force due to the brakes is required to stop the train within 200 ft? (Neglect other frictions.)

10–17. In the apparatus of Fig. 164, $CD = 5$ ft, $CE = 50$ ft, $W = 9000$ lb, $P = 40$ lb. Find the velocity of the system after 4 sec.

10–18. A bullet weighing 2 oz and going at the rate of 1000 fps penetrates 15 in. into an oak post. Find the average resistance to penetration offered by the post.

10–19. In Example 1, Art. 75, what will be the tension in the cord over the pulley?

10–20. If in Fig. 161 A is 50 lb and B 30 lb, and A is moving upward with velocity of 20 fps, how far and how long will it move before coming to rest?

10–21. A platform sustaining a weight of 120 lb is descending with a uniform acceleration of 4 fps^2. Find the pressure of the weight on the platform.

10–22. Find the pressure of the weight on the platform in Problem 10–21 if the platform is ascending:

(*a*) With a uniform velocity of 9 fps,

(*b*) With a uniform acceleration of 10 fps^2.

10–23. In an inclined plane the slope is 20 ft, and the height is 4 ft. A body weighing 100 lb rests on the incline. Find:

(*a*) The acceleration with which the body will slide down the plane if the friction is 5 lb,

(*b*) The time required to slide 10 ft.

10–24. A body is projected up a smooth incline rising 2 ft for each 50 ft of slope with an initial velocity of 200 fps. Find its velocity at the end of 6 sec. How far will the body travel up the incline in 10 sec?

10–25. In what time will the body of Problem 10–24 come to the highest point it will reach on the incline?

10–26. A car rolls down an incline 80 ft long and having an inclination of 45° to the horizontal. Find the velocity of the car at the foot and the time for its descent if friction is neglected.

10–27. An elevator loaded with passengers weighs 2450 lb. It is counterbalanced with weights of 950 lb. With what acceleration would the elevator descend if it were free to fall and there were no friction on the pulleys or guides?

10–28. A N. Y., N. H. & H. electric locomotive, weighing 200 tons and hauling a 200-ton train, exerts an average tractive force of 10,000 lb. Neglecting friction, what time will be required to get up a speed of 60 mph from rest (*a*) on a level track? (*b*) on a 1 per cent upgrade?

10–29. If the locomotive of Problem 10–28 is required to get up a speed of 60 mph in 200 sec, what is the maximum trailing load it can haul?

10–30. If the train of Problem 10–28 starts up a 3 per cent grade at a speed of 55 mph, what will be its speed at the end of a half mile?

10–31. An automobile weighing 3200 lb accelerates from rest to 25 mph in 8 sec. Find (*a*) the acceleration in feet per second per second, (*b*) the distance traveled in the 8 sec, (*c*) the tractive force required (in excess of that required to overcome friction and the inertia of the rotating parts).

10–32. Find the distance in which a 3200-lb automobile traveling at 40 mph can be stopped if the effective braking force is 60 per cent of the car's weight. If the driver's reaction time (to move his foot from accelerator to brake pedal after he has the impulse to do so) is 1 sec, how much greater distance will be required?

10–33. If the driver of a car weighing 3600 lb, traveling at the rate of 60 mph wants to stop at a signal, how far ahead must he start to act? Assume his reaction time is 1 sec (before his foot obeys the impulse to move from gas to brake pedal) and that the effective braking force is 0.6 the weight of the car.

10–34. An automobile accelerates from 30 mph to 50 mph in 10 sec. Its weight is 3000 lb. Find the tractive force required (neglecting friction and inertia of rotation).

76. Force and acceleration in rotary motion. Torque. In the preceding paragraphs of this chapter, it has been shown that inertia is the property of all matter by reason of which it offers resistance to acceleration; that inertia depends solely upon mass; and that the fundamental relations in motion of translation are expressed by the equation, *force* = *mass* × *acceleration*.

Because of the nature of motion of rotation, certain other factors must be taken into account. Thus, the ability of a force to produce rotation does not depend upon its amount alone but also upon its moment arm. A small force having a long moment arm

will produce as much *torque* as a greater force having a correspondingly short arm.

The resistance offered by a body to acceleration in rotation (i.e., *its rotational inertia*) does not depend upon mass alone but *also upon the way in which the mass is distributed with respect to the axis of rotation.* **The inertia reactions of masses farther out from the axis have greater moment arms and therefore react with greater effect in opposing change in velocity.** Thus it is a matter of common experience that a wheel with its mass mainly in its rim offers much greater resistance to starting, and also stopping, than one which weighs exactly the same, but which has its material concentrated near its axle.

Then, also, in rotation, the *linear acceleration* given the particles of the body *must be greater in proportion to their distances out from the axis of rotation.*

If however, in the fundamental idea of force = mass $\times$ acceleration, torque is substituted for force; angular acceleration (which is constant for all portions of the body) for linear; and a new unit of rotational inertia for mass; then the relation will be:

Torque = rotational inertia $\times$ angular acceleration

Rotational inertia is commonly called moment of inertia (see next Article). Letting T represent torque in pound-feet, α represent angular acceleration in radians per second per second, and I represent moment of inertia. The equation will be

$$T = I\alpha \tag{22}$$

77. Moment of inertia. The resistance which a body offers to angular acceleration (i.e., its rotational inertia) is commonly called its moment of inertia. *Unit moment of inertia* is the moment of inertia of a body to which unit moment of force will impart unit angular acceleration.

It may be shown that the moment of inertia of a particle is equal to the mass of the particle m, multiplied by the square of its distance r from the axis of rotation or mr^2. The moment of inertia I of a body equals the sum of the moments of inertia of its separate particles, or

$$I = \sum mr^2 \tag{23}$$

To compute the moment of inertia of a body, therefore, it is

necessary to multiply the mass of each of its particles by the square of its distance from the axis and then to find the sum of all such products, by the method of calculus. This is beyond the scope of this book and the reader who is interested is referred to more advanced treatises and to tables of moments of inertia, etc.

It may be noted that the expression for the moment of inertia of a body is similar in form to that for the moment of inertia of a sectional area; see Art. 41 and Art. 129.

78. Radius of gyration. Attention has already been called to the obvious fact that in rotatory motion portions of the rotating body at different distances from the axis have different linear velocities. It is evident upon consideration, however, that there must be a point in every rotating body which will move with such a velocity that if the entire mass of the body could be concentrated at this point, the inertia resistance of the body would be unchanged. **The radius of gyration is the distance from the axis of rotation to the point at which the entire mass of the rotating body can be considered concentrated to give the same inertia reaction.** It is represented by the symbol k.

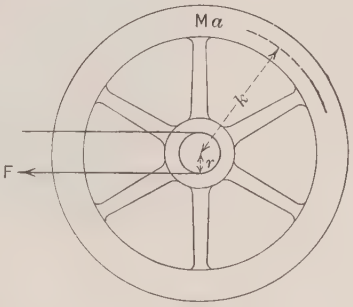


Fig. 168.

In rotatory motion we may, therefore, when more convenient, consider the entire mass of the body as having the same linear velocity as a point situated a distance equal to the radius of gyration from the axis of rotation. Then the equation expressing these relations is:

$$I\ (=\Sigma mr^2) = mk^2 \quad \text{and} \quad k = \sqrt{\frac{I}{m}} \tag{24}$$

The value of the radius of gyration for various common forms as disks, cylinders, etc., is given in most practical handbooks. In flywheels having a heavy rim it is customary to neglect the mass of the hub and spokes and regard the radius of gyration as the distance from the axis to middle of the rim as k, Fig. 168.

79. Flywheels and shafts. The preceding principles can be applied to rotating machinery as follows: Referring to Fig. 168 let

F be the force causing rotation and r the moment arm of this force about the center of the shaft; then Fr = the torque. Also let the mass of the rim be considered concentrated at a distance k,

Fig. 169. Flywheel for a generator unit.

the radius of gyration, from the center of the shaft. Then the value of $I\ (= \sum mr^2) = mk^2$. From Art. 76 $T = I\alpha$. Combining these equalities gives:

$$Fr = mk^2\alpha \tag{25}$$

for *acceleration in radians per second per second.* If, instead of angular acceleration, the linear acceleration, a, at radius k is used, since $\alpha = a/r$ or $= a/k$ in this case, the equation can be written:

$$Fr = mk^2 \times \frac{a}{k} = mak \tag{25a}$$

for *linear acceleration in feet per second per second* at radius k.

The same equation may be derived in slightly different manner. If m is the mass of the rim and a is the linear acceleration at distance k from the axis,

ma is the measure of the inertia reaction of the rim. Then the torque which *produces* rotation $= Fr$, and the torque which *resists* rotation $= ma$ multiplied by its arm, k, $= mak$.

Again, if it is desired to use *acceleration in revolutions per second per second*, then, since radians per sec^2 = rps$^2 \times 2\pi$, the equation can be written:

$$Fr = mk^2\, 2\pi\ (\text{rps}^2) \tag{26}$$

Equation (25) above is the fundamental equation for the relation of torque, mass, and acceleration for rotating bodies where the radius of gyration, k, is known. Equation (26) is commonly used for practical computations. Force F is usually the difference in tension between the tight and loose sides of the driving belt, and r is the radius of the belt pulley.

It must be remembered that in all the foregoing equations, $m = W/g$.

Example.

A flywheel, as Fig. 170, whose rim weighs 8000 lb is started from rest by a pull of 1000 lb acting at the circumference of a belt pulley 12 in. in diameter The outside diameter of the flywheel is 6 ft 6 in., thickness of rim 6 in. How long will it take to reach a speed of 120 rpm? (Neglect friction.)

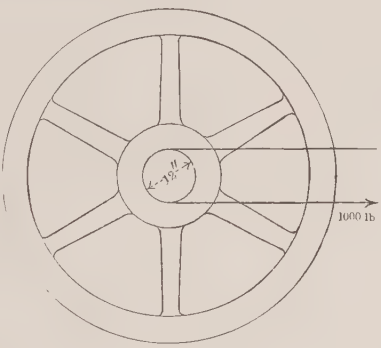


Fig. 170.

Assume the radius of gyration to be to the middle of the rim, neglecting the relatively small amount in the spokes and hub.

Solution. The radius of gyration $= 6 \div 2 = 3$ ft. The mass of the rim $= \dfrac{8000}{32} = \dfrac{W}{g}$. The moment arm, $r = \dfrac{1}{2}$ ft.

From (26) $\qquad 1000 \times \dfrac{1}{2} = \dfrac{8000}{32} \times 3 \times 3 \times 2\pi \times (\text{rps}^2)$

From which $\qquad (\text{rps}^2) = 0.035$ the acceleration required

Since $\qquad 120$ rpm $= 2$ rps to be gained,

$$\text{Time required} = \frac{2}{0.035} = 57 \text{ sec, approximately.} \quad \textit{Ans.}$$

Or, using the equation (25), $Fr = mk^2\alpha$

$$1000 \times \frac{1}{2} = \frac{8000}{32} \times 3 \times 3 \times \alpha$$

$$\alpha = \frac{500 \times 32}{8000 \times 9} = 0.22 \text{ radians per sec}^2$$

and $0.22 \div 2\pi = 0.035$ rps^2, as before

PROBLEMS

10–35. What pull in a belt running on a belt pulley 20 in. in diameter will be required to increase the speed of a flywheel from 120 rpm to 180 rpm in 10 sec if the weight of the flywheel is 1600 lb and its radius from axis to the center of the rim is 2 ft?

10–36. A force of 10 lb is acting on a flywheel at a distance of 3 in. from the axis for a time of 5 sec. Find the speed and the total number of turns, starting from rest, made by the wheel if it weighs 64 lb and is of such a size and shape that $k = 9$ in.

10–37. A uniform circular disk 4 ft in diameter weighing 1000 lb is caused to rotate from rest by a force of 100 lb acting at its circumference. What will be the linear velocity of a point on the circumference at end of $1\frac{1}{2}$ min?

Note. For a uniform circular disk, axis at center k = approximately $\frac{7}{10}$ of the radius.

10–38. What force acting at the rim of the disk of Problem 10–37 will be required to change its speed from 300 rpm to 120 rpm in 20 sec?

80. Centrifugal force. Because of their inertia, bodies set in motion tend to continue moving in a straight line. Thus a body moving from the positions M_1, M_2, etc., Fig. 171, tends to go along the straight paths, M_1, N_1, M_2N_2, etc. To pull it out of a straight path, and *compel it to move in a circle, requires a force acting constantly toward the center.* This force toward the center, which is necessary to make a body rotate about this center, is called the **centripetal force.** The equal and opposite reaction to the centripetal force is called **centrifugal force.** It is this "centrifugal force" which sometimes causes rapidly rotated flywheels, emery wheels, etc., to burst. The rotating parts of all machinery must be carefully

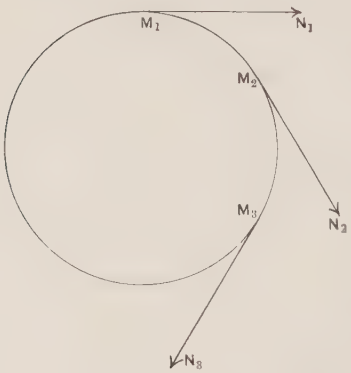


FIG. 171.

balanced about the axis, otherwise centrifugal force may cause injurious wear at the bearings, jarring, etc.

It should be noted that centrifugal force is not a force in the usual sense of the word, *but is a reaction due to inertia.*

It can be shown (see note at end of this article) that the force that must act toward the center to keep a body moving at a uniform velocity in a circular path *depends upon the mass of the body, upon the square of the velocity, and upon the size of the circle*; and that the relationship between these quantities is expressed by the equation

$$\text{Centrifugal force} = \text{centripetal force} = \frac{mv^2}{r} \qquad (27)$$

Example.

What is the centrifugal force of each pound of weight of the rim of a flywheel 6 ft in diameter, making 400 rpm?

Solution.

$$\text{Velocity} = \frac{6 \times \pi \times 400}{60} = 126 \text{ fps, approx.}$$

$$\text{Therefore, centrifugal force} = \frac{1 \times 126 \times 126}{32 \times 3} = 165 \text{ lb} \quad \textit{Ans.}$$

Note. Suppose a body of mass m to be moving with uniform velocity v around the circle ACE, Fig. 172. Let t be the time required to move over the arc AC. In the same time, the body has been pulled out of its straight path an amount equal to the distance BC by a force which, as the path is circular, must everywhere be directed toward the center O. If a be the acceleration which this force will impart to the mass m, $BC = at^2/2$. If the time t be sufficiently short, arc AC will be so short that it may be regarded as a straight line: figures ACD and ACE will then be similar triangles. Therefore,

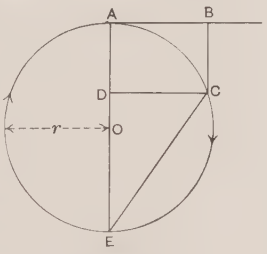


FIG. 172.

$$\frac{AD}{AC} = \frac{AC}{AE} \qquad (1)$$

$$(AC)^2 = AD \times AE \qquad (2)$$

But $AC = vt$ (distance moved in uniform motion)

$$AD = BC = \frac{at^2}{2} \text{ (distance moved, accelerated motion)}$$

And $AE = 2r$, r being the radius of the circle

Substituting these values in equation (2) and solving, we have $a = v^2/r$, or the acceleration imparted by the force acting toward the center of the circle equals the square of the velocity divided by the radius of the path. And, since force equals mass multiplied by acceleration, we have

$$\text{Centripetal force} = \frac{mv^2}{r}$$

PROBLEMS

10–39. In a machine running at 1800 rpm there is an unbalanced weight of 1 lb at a radius of 1 ft. Find the pull on the bearings due to centrifugal force.

10–40. A bicycle and rider weighing together 200 lb run around a curve of radius 50 ft at a rate of 12 mph. Find the force with which the tire must stick to the track to prevent slipping.

10–41. How many revolutions per minute will suffice to break a string of a tensile strength of 10 lb if a body weighing 5 lb be swung round at the end of 6 ft of the string?

10–42. A train of 200 tons weight is rounding a curve of $\frac{1}{2}$-mile radius with a velocity of 30 mph. What is the horizontal pressure on the rails?

10–43. What should be the elevation of the outer rail for a standard gauge track of 4 ft $8\frac{1}{2}$ in. for the train of Problem 10–42?

10–44. Find the speed at which a simple Watt governor runs when the arm makes an angle of 30° with the vertical if the length of the arm from the center of the pin to the center of the ball is 18 in.

Note. The steam governor consists essentially of two balls, each at the end of a rod or arm hinged by a pin to the top of a vertical rotating shaft on the engine. Because of centrifugal force the balls tend to fly out as the speed increases, causing a greater angle between arm and shaft, which movement operates a lever to decrease the inlet of steam. As the speed decreases, and consequently the centrifugal force lessens, the balls swing back in on their hinged arms, which movement opens the steam port again. Each ball is subject to the force of gravity and to the centrifugal force.

10–45. The rim of a cast-iron flywheel has a mean radius of 2 ft. Its section is 6 in. broad and 4 in. thick. The flywheel is revolving at 180 rpm. What is the centrifugal force per inch length of rim?

10–46. A pulley weighing 25 lb is making 2400 rpm. If its center of gravity is $\frac{1}{16}$ in. from the axis of rotation, what is the centrifugal force producing pressure against the bearing and resulting wear?

SUMMARY

Forces causing motion of a body **are proportional to the accelerations caused.**

$$\frac{F_1}{F_2} = \frac{a_1}{a_2} \qquad (17)$$

A force causing motion of a body has the same ratio to the acceleration caused as has the weight of the body to the acceleration of gravity.

$$\frac{F}{a} = \frac{W}{g} \tag{18}$$

The relation between force, mass, and acceleration is:

$$F \propto ma \tag{19}$$

Mass is the quantity of matter. **The unit of mass** is represented by m, and in the usual system $= W/g$. The proportion of formula (19) expressed in the engineering or English gravitational system of units is:

$$F = ma \quad \text{or} \quad F = \frac{W}{g}a \tag{20}$$

The acceleration of a body on a smooth incline can be shown to be expressed by the equation:

$$a = g \sin \delta \tag{21}$$

The procedure in solving acceleration problems is:

1. Determine the direction of the change in velocity.
2. Construct an "acceleration diagram," showing all actions in the direction of the change in velocity, and all reactions in the opposite directions. Always include among the latter the one due to inertia.
3. Substitute the values for these actions and reactions in the general equation and solve.

The general equation for accelerated motion is: The sum of all actions in the direction of the change of velocity equals the sum of all reactions in the opposite direction.

In the simple cases the action in one direction is the force causing acceleration (positive or negative), whereas the reactions in the opposite direction are friction and the inertia reaction.

The inertia reaction $= ma = \frac{W}{g}a$

In angular acceleration, the force applied times its moment arm causes a torque, which is resisted by the mass inertia reaction of the body. The inertia reactions of masses farther out from the axis of rotation have greater moment arms, and therefore react with greater effect in opposing a change in velocity. The resistance which a body offers to angular acceleration is called its **moment of inertia.**

The radius of gyration is the distance from the axis of rotation to the point at which the entire mass of the rotating mass of the body could be concentrated to give the same inertia reaction. The equations for the foregoing relations are:

$$T = I\alpha \tag{22}$$

$$I = \Sigma mr^2 \tag{23}$$

$$I = mk^2 \quad \text{and} \quad k = \sqrt{\frac{I}{m}} \tag{24}$$

$$Fr = mk^2\alpha \tag{25}$$

$$Fr = mak \tag{25a}$$

$$Fr = mk^2\, 2\,\pi\ (\text{rps}^2) \tag{26}$$

Centrifugal force is equal and opposite to centripetal force. **Centripetal force** is that force which acts toward the center of rotation to keep a rotating body moving in a circle instead of in a straight line. It depends upon mass, velocity, and the radius of the circle. The equation is:

$$\text{Centrifugal force} = \text{centripetal force} = \frac{mv^2}{r} \tag{27}$$

CHAPTER XI

WORK AND POWER

Chapter XI treats of some of the most practical applications of the subject of mechanics. These include the meaning of work in the technical sense of force times displacement; and power, which is a measure of the rate at which work is accomplished. The instruments for measurement of power and the computation of power are also taken up in this chapter, preliminary to the study of machines in Chapter XIV.

81. Work. When a force causes a body to change its position against a resistance opposing such change, *work* is done on the body. Thus work is done upon a body when the body is lifted against the force of gravity; when the body is moved against the resistance offered by friction; or when the velocity of the body is increased or decreased against the reaction offered by inertia.

Two factors must always be present when work is done; viz., *force* (equal to the opposing resistance), and *displacement.* **The measure of the work done is the product of the force and the displacement in the direction of the force.** Then letting s represent the displacement in the direction of the force, F, and W the work done,

$$W = Fs \tag{28}$$

Where the displacement is not in the line of action of the force, either the component of the displacement in the direction of the force must be found, or the component of the force in the direction of the displacement. The work is then equal to the *product of the force and the component of the displacement in the direction of the force*; or, it is equal to the *product of the displacement by the component of the force in the direction of the displacement.* Thus, suppose a body weighing 100 lb is to be pushed 10 ft up a smooth plane rising 30° to the horizontal. The resistance here is due to the vertical gravity pull of 100 lb. The displacement in the direction of this force is 10 sin 30° or 5 ft. Hence the work done is $100 \times 5 = 500$ ft-lb. Or, the component of the force in the

direction of displacement is 100 sin 30° = 50 lb, and the work done is therefore 50 × 10 = 500 ft-lb.

Or, suppose a body weighing 200 lb resting upon a horizontal table is pushed 3 ft along the table by a force of 60 lb acting at an angle of 20° with the horizontal. The force in the direction of displacement is here 60 × cos 20° = 56.4 lb. The work is, therefore, 56.4 × 3 = 169.2 ft-lb. The weight of the body and the component of the force perpendicular to the table do no work as they produce no displacement in their directions.

82. Units of work. Work may be expressed in different units, dependent upon the units used in defining the force and the displacement. Thus, if force is measured in *pounds* and displacement in *inches*, the work will be expressed in *inch-pounds*. Other units are foot-pounds, foot-tons, etc.

The unit of work commonly used is the **foot-pound.**

A foot-pound is the work done when a force of one pound acts through a distance of one foot in the direction of the force; or when *an equivalent amount of work* is done, as when a force of 2 lb acts through a distance of $\frac{1}{2}$ ft, a quarter pound acts through a distance of 4 ft, etc.

In the metric system **an erg is the work done by a force of one dyne acting through a distance of one centimeter.** The erg is thus a very small amount of work; for convenience of expression, therefore, the **joule,** or 10^7 **ergs,** is often used.

83. Work diagrams. Since work equals the product of force and distance, it may often be conveniently represented by the area of a diagram as Fig. 173. Thus, suppose a constant force of 100 lb acts through a distance of 16 in.; the work done = 100 × 16 = 1600 in-lb. By taking any convenient scales, as 1 in. vertical = 20 lb, and 1 in. horizontal = 4 in., the 100-lb force may be represented by a vertical line 5 in. long and the 16-in. distance by a horizontal line 4 in. long, both drawn from the same point. With these scales an area of 1 sq in. = 20 × 4 = 80 in-lb. The area of the parallelogram drawn on the two lines as adjacent sides = 5 × 4 = 20 sq in., and therefore represents 20 × 80 = 1600 in-lb of work.

If the force doing work is not constant, the work done may still be represented by an area. Thus, suppose Fig. 174 represents the conditions in a steam-engine cylinder where, as is usually the case,

the steam pressure is not constant during the entire stroke. The vertical height of the diagram at any point represents the steam pressure in pounds per square inch at that point in the stroke.

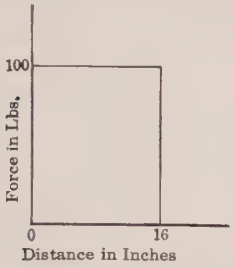


FIG. 173. Area represents work in inch-pounds.

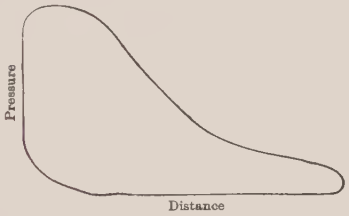


FIG. 174. Indicator card.

The area of the figure (*average height* × *length*) therefore represents, in terms of the scale to which it is drawn, the product of the average steam pressure times length of stroke, or the work done by the steam on each square inch of the piston as it moves forward the length of one stroke.

Suppose the area of the diagram is 2.2 sq in. and suppose $\frac{1}{10}$ in. = 1 in. is the horizontal scale, and 1 in. = 30 lb per square inch is the vertical scale. Then each square inch of the diagram represents $10 \times 30 = 300$ in-lb, and the whole diagram represents $2.2 \times 300 = 660$ in-lb or 55 ft-lb.

The area of irregular figures such as the above may be found approximately by measuring the height at 10 or more points at equal intervals along its entire length and taking the average

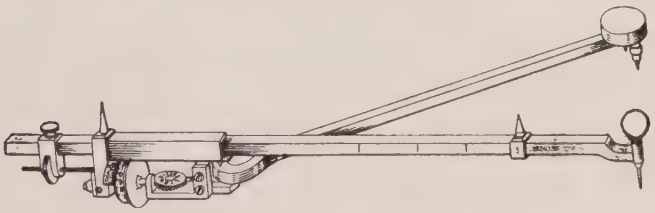


FIG. 175. Planimeter.

of these measured values as the average height. Average height × length = area. A more accurate method is to use for measuring areas an instrument known as a **planimeter.** See Fig. 175.

Work diagrams similar to Fig. 174, taken from engines in actual

use, are known as **indicator cards**; instruments for taking indicator cards are known as **steam-engine indicators.** In practice, steam-engine indicators are " calibrated " so that the values of the vertical and horizontal scales of the diagram in terms of actual steam pressure and length of stroke are known.

84. Power. In the previous discussion of work it will be noticed that the *time* required for the force to act through a given distance *was not considered*; the total work is the same whether performed quickly or slowly. But a machine which will do a given amount of work in a short time is said to be more powerful than one that requires greater time in which to do the same work.

Power is therefore defined as the rate of doing work. Power equals work divided by time. Letting P represent power,

$$P = \frac{W}{t} = \frac{Fs}{t} \tag{29}$$

85. The measurement of power. Power may be expressed in any units which specify the work done and the time in which it is performed. The customary units of power are, however, the *horsepower* and the *watt.*

Work is done at the rate of **one horsepower** (abbreviated hp) **when 33,000 ft-lb of work are done in one minute.** Therefore

$$\text{Horsepower} = \frac{\text{foot-pounds in one minute}}{33{,}000} \tag{29a}$$

Also $$\text{Horsepower} = \frac{\text{foot-pounds per second}}{550}$$

The watt is the electrical unit of power. It is equal to the product of the electrical pressure in volts multiplied by the current in amperes which it is producing, i.e.,

$$\text{Watts} = \text{volts} \times \text{amperes} \tag{30}$$

The relation between horsepower and watts is

$$1 \text{ hp} = 746 \text{ watts} \tag{30a}$$

$$1 \text{ kilowatt} = 1000 \text{ watts} = \tfrac{1000}{746} \text{ hp} = \text{about } 1\tfrac{1}{3} \text{ hp}$$

The horsepower of machines can be found by the application of the above principles. There are several cases in common use for which special formulas have been derived for convenience. *Care*

must be taken to avoid confusion in the use of formulas, and the student is advised to understand thoroughly and apply the underlying principles to the solution of problems before substituting values (more or less blindly) in the formulas.

The *horsepower* of a belt is found from the formula:

$$\text{Horsepower} = \frac{(T_1 - T_2)\, v}{33{,}000} \tag{31}$$

in which T_1 is the tension in the tight side of belt, T_2 the tension in the slack side, and v the velocity of the belt in feet per minute. $T_1 - T_2$ = the force (F) acting, and v = the distance through which it acts in a minute. Their product gives foot-pounds per minute, which divided by 33,000 gives horsepower.

The belt speed is found from pulley speed by the formulas in Art. 56. Letting n = the number of revolutions per minute and D = the diameter of the pulley *in feet*, the foregoing equation becomes:

$$\text{Horsepower} = \frac{(T_1 - T_2) \times \pi \times D \times n}{33{,}000} = \frac{F\pi Dn}{33{,}000} \tag{32}$$

If the torque of the pulley on a belt or rope is known, that value, T, may be used in the preceding equation. Since $D = 2\,r$, the equation can be written

$$\text{Horsepower} = \frac{F\ 2\ \pi rn}{33{,}000}$$

and, since $Fr = T$, this may be substituted giving:

$$\text{Horsepower} = \frac{2\ \pi nT}{33{,}000} \tag{33}$$

The horsepower of a steam engine is found from the force on the piston and the displacement of the piston. Thus let p be the pressure and A the area of the piston, then $p \times A$ = the force in pounds. Also let L = the length of the stroke in feet and n the number of strokes per minute, then $L \times n$ = the distance through which the force acts.

Therefore $$\text{Horsepower} = \frac{pLAn}{33{,}000} \tag{34}$$

Horsepower lost because of friction in bearings is discussed in Art. 101.

Example 1.

How much work is done in 2 min in lifting a weight of 500 lb by a rope wound around a 10-in. diameter drum turning at 12 rpm?

Solution. Force expended, neglecting friction, equals 500 lb.

Distance moved in 2 min $= \pi \times \frac{10}{12} \times 12 \times 2 = 62.8$ ft

$W = Fs = 500 \times 62.8 = 31{,}400$ ft-lb, work done in 2 min *Ans.*

Example 2.

Find the horsepower required in the preceding problem, friction neglected.

Solution.

$$\text{Horsepower} = \frac{\text{foot-pounds per minute}}{33{,}000} = \frac{31{,}400}{2 \times 33{,}000} = 0.48 \text{ hp} \quad \textit{Ans.}$$

Example 3.

Given a belt running at a speed of 2800 fpm with an effective tension of 16 lb. How much work is it doing per hour? What is the power transmitted?

Solution.

$$W = Fs = 16 \times 2800 \times 60 = 2{,}688{,}000 \text{ ft-lb in an hour}$$

$$\text{Horsepower} = \frac{16 \times 2800}{33{,}000} = 1.36 \text{ hp} \quad \textit{Ans.}$$

Example 4.

Compute the horsepower developed by a single acting engine if the piston diameter = 4 in.; length of stroke = 6 in.; pressure per square inch = 30 lb; speed = 200 rpm.

Solution.

$$\text{Force} = 30 \times 4 \times 4 \times 0.7854 = 377.0 \text{ lb}$$

Distance $= \frac{6}{12} \times 200 = 100$ ft (power on one stroke per revolution in single acting engine)

$$\text{Horsepower} = \frac{\text{foot-pounds per minute}}{33{,}000} = \frac{377 \times 100}{33{,}000} = 1.14 \text{ hp} \quad \textit{Ans.}$$

86. Dynamometers. Instruments for measuring power are commonly called *dynamometers*. These are, generally speaking, of two classes: (1) *absorption dynamometers* which absorb the power received, transforming it into heat; and (2) *transmission dynamometers* which absorb only power enough to operate their mechanism and transmit the remainder.

Dynamometers of the first type are used where the output of a

source of power is to be measured, as for example, the power delivered at the belt pulley of a motor, engine, etc. This is termed the **brake horsepower** of the motor or engine.

Those of the second type are inserted *between* the source of power and the machine being driven to determine the power delivered to any particular belt, shaft, etc., in the transmission line. Transmission dynamometers thus enable us to determine the power lost at any stage in the power transmission, through friction, belt slip, etc.

The simplest and most commonly used form of absorption dynamometer is the **Prony brake.** This may be of several different forms, all of which are based upon the same fundamental principle. Figure 176 shows a common form, where *A* indicates the belt pulley of a motor, engine, etc. This is surrounded by the brake band *B*, which may be tightened as desired by means of the screw at *C*. As the pulley revolves in the direction of the arrow, friction tends to carry the band around also, but this tendency is counteracted by a force *W* acting on a lever arm *L* attached to the band.

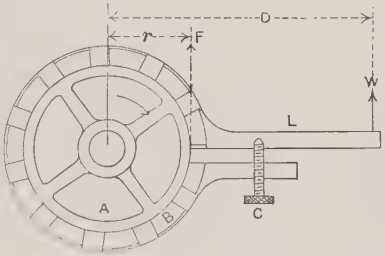


FIG. 176. Prony brake.

If *F* represents the force of friction,

$$F \times r = W \times D$$

and

$$F = \frac{W \times D}{r}$$

The work in foot-pounds absorbed by the brake in one revolution is equal to the friction in pounds times the distance in feet through which it is overcome, or

$$F \times \text{circumference of pulley}$$

and the power absorbed is, therefore,

$$\frac{F \times \text{circumference of pulley} \times \text{revolutions per minute}}{33{,}000}$$

which is expressed by equation (32).

The friction brake evidently transforms the mechanical energy delivered into heat. The great amount of heat thus generated usually makes it necessary to use a cooling device with the brake. Pulleys with hollow rims which may be filled with water, or hollow brake straps through which a stream of water is kept flowing, are, therefore, often used.

Allowance must be made for the brake *tare* or *zero reading* due to the weight of the brake arm, part of which rests on the scales. One method to get a correct reading, one which does not include

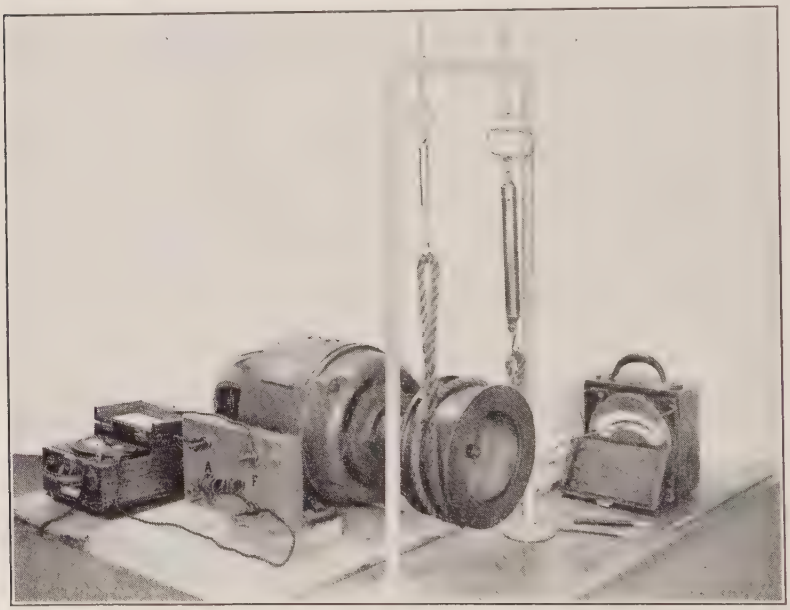


Fig. 177. Brake horsepower test of a motor.

also an error due to friction or " grab " of the brake blocks, is to loosen the brake as much as possible, and turn the pulley first forward, and then in reverse, and average the readings on the scale for the tare. Another method is to loosen the brake, move the brake arm by hand, up and down and sideways, to free it as far as possible from the pulley, and take the average of several readings.

The **rope form** of the Prony brake is also in common use, particularly on small and medium-sized electric motors. See Figs. 177 and 178. The rope is wound once around the pulley and each

end fastened to a spring balance. By means of a turnbuckle the rope can be tightened to vary the load. The difference in tension between the two balances gives the net force, and the diameter and speed give the means to determine the distance through which it acts. If desired formula (32) for belt horsepower in the preceding article may be used. It is usual practice to consider that the force acts at the center of the rope; therefore, in computing the circumferential distance, the diameter of the pulley should be increased by the diameter of the rope (one-half on each face of pulley).

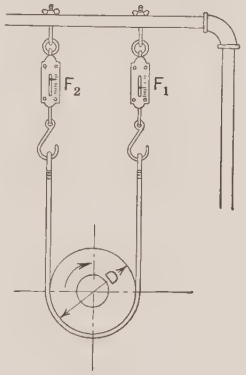


FIG. 178. Rope form of Prony brake.

The object in all **transmission dynamometers** is to absorb as little power as possible in the dynamometer itself and to measure the amount transmitted. One type is shown in Fig. 179. This shows a simple *dynamometer coupling* in which connection between the two parts of a shaft is made by the springs *SSS* instead of bolts or pins, as in the ordinary rigid coupling. Both plates *A* and *B* are keyed to the shaft. As the side *B* is driven, the torque is communicated to *A* through the springs which are thereby elongated. If, now, the *constant of the springs*

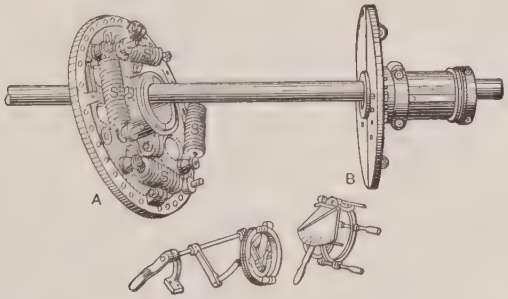


FIG. 179. Transmission dynamometer.

is known, also the *amount they are extended*, and *the speed* in revolutions per minute, the horsepower transmitted by the shaft may be readily computed.

A type of transmission dynamometer for use with a belt is shown in Fig. 180. A vertical plate carrying four pulleys upon

which the driving belt runs is balanced about a pivot at C. Power is then turned on, and the plate again balanced. If G = weight to balance,

$$Gb = Pa - P'a$$

(P and P' on the right pass through C and therefore have no moment.)

Therefore, $$P - P' = \frac{Gb}{a}$$

Work per minute $= (P - P')\,v$

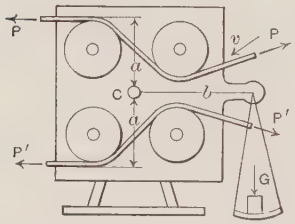


FIG. 180. Belt dynamometer.

when v is the velocity of the belt in feet per minute.

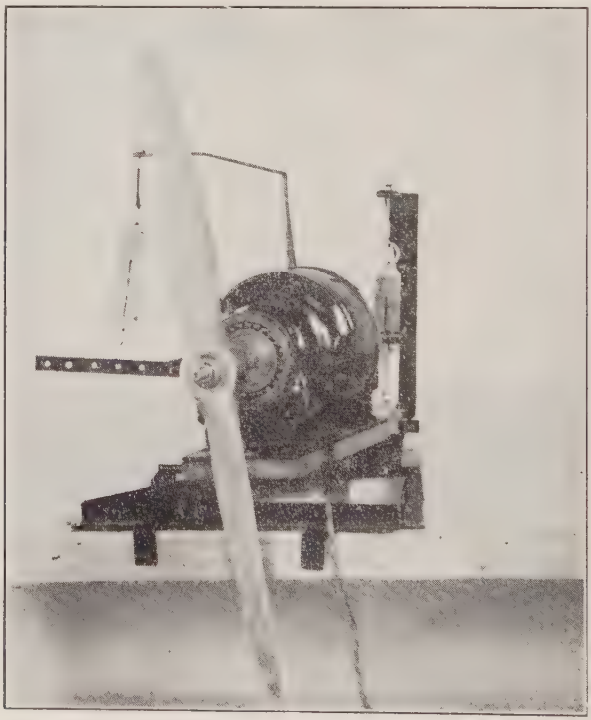


FIG. 181. A cradled motor dynamometer, rigged for test of a model airplane propeller.

For a more detailed discussion of particular types of transmission dynamometers, the student is referred to engineering textbooks, trade periodicals, etc.

A unique type is the **cradled motor dynamometer.** It is particularly useful in making efficiency tests of motor-driven machines. The motor is cradled in ball bearings on the frame, the stator and frame being free to turn on the armature or rotor shaft. When the motor is running there is a reaction on the stator, equivalent to the torque required for driving the load. The torque is determined by measuring the force at the end of an arm of known length attached to the frame. The power output of the motor, and therefore power input of the machine being driven, can be computed from the torque and the motor speed. This method has the advantage of being quite direct, and it eliminates some of the losses and inaccuracies of other types. See Fig. 181.

PROBLEMS

11–1. A car weighing 3000 lb and containing 2 tons of coal is hoisted up the shaft of a mine 400 ft deep. What is the work done?

11–2. The weight of a trolley car is 8 tons. The average resistance to motion due to friction, etc., is one-sixteenth of the weight. What work is done in a run of 4 miles on a level track? What power is required if the run is made in 20 min?

11–3. James Watt found that an ordinary English dray-horse could travel an average of $2\frac{1}{2}$ mph and raise 150 lb by means of a rope and pulley. Find the work done per minute.

11–4. How much work is accomplished by raising a machine weighing 686 lb to a floor 69 ft up in a building? How much power if the time is 5 min?

11–5. Four men lift a concrete slab 30 in. $\times$ 24 in. $\times$ 4 in. in size through a distance of 4 ft. If the concrete weighs 145 lb per cu ft, how much work does each man do?

11–6. A ladder 28 ft long rests against a wall 24 ft high. How much useful work is done by a man in carrying 50 lb of brick to the top of the wall? How much if he also carries the brick 20 ft along the horizontal top of the wall?

11–7. If the man in Problem 11–6 weighs 160 lb how much necessary but useless work does he do when going up the ladder? How much when coming down?

11–8. A man weighing 180 lb walks up stairs, a height of 20 ft, in $\frac{1}{2}$ min. At what rate does he work?

11–9. A load is hauled along a level floor by means of a rope inclined 30° to the horizontal. If the pull in the rope is 100 lb and the body is moved 10 ft, compute the work done.

11–10. A body 4 ft $\times$ 3 ft $\times$ 2 ft rests upon the 2 $\times$ 3 face. The body weighs 150 lb. Compute the work done in turning it over on the 2-ft edge.

11–11. An engine hauls a car 10,000 ft along the track. The pull exerted by the engine upon the car in the beginning was 3500 lb and as recorded every

1000 ft was as follows: 3500, 3250, 2800, 3450, 3700, 3600, 3550, 3300, 2950, and 2750 lb. Construct a "work diagram" for above data. Determine the area of the diagram and from this the total work done on the car.

11–12. A punch exerts an average pressure of 50 tons in punching a hole through a plate $1\frac{1}{8}$ in. thick. Compute the work done.

11–13. A well 6 ft in diameter and 30 ft deep is dug. Compute the work done in raising the material if 1 cu yd weighs 3000 lb.

11–14. Find the horsepower required to fill a tank with water in an hour, if the tank is $10 \times 12 \times 3$ ft in size and the water must be raised 50 ft. (Weight of water = 62.5 lb per cu ft.)

11–15. Find the horsepower of the engine that should be used for raising coal from a pit 200 ft deep, if the average daily yield is 1000 long tons, and the engine works 15 hours per day. (A long ton = 2240 lb.)

11–16. A train weighs 220 tons. The resistance to motion is 20 lb per ton on a level. Find the horsepower of an engine which can just keep the train going 30 mph up a grade of 10 ft rise per mile of track.

11–17. A 5-hp engine is used to pump water from a shaft 100 ft deep. How many cubic feet will it raise in 24 hr? (Cubic foot of water weighs 62.5 lb.)

11–18. What horsepower is required to supply 1000 families with 60 gal water each per day of 10 hr, if the water is pumped to a height of 200 ft? (Gallon of water weighs $8\frac{1}{3}$ lb.)

11–19. Find the work done against a brake which gives a frictional resistance of 15 lb on the rim of a wheel of 2 ft radius, while the wheel makes 50 revolutions.

11–20. Find the horsepower used in the preceding problem if the 50 revolutions required 20 sec time.

11–21. Compute the horsepower developed by an engine under the following conditions: Diameter of piston 4 in.; length of stroke 5 in.; mean effective pressure 42 lb per sq in.; number of revolutions 275 per min.

11–22. If the driving tension of a belt is 45 lb, the back tension on the loose side of belt 4 lb, the pulley diameter 18 in, and the pulley speed 240 rpm, find the horsepower delivered.

11–23. If a Prony brake (rope type) on a 10-in. motor pulley running at 800 rpm reads 39 lb on one balance and 1 lb on the other, find the brake horsepower. The rope is 1 in. in diameter.

11–24. Given a belt running at a speed of $\frac{1}{2}$ mile per minute with a net pull of 45 lb. Find the power.

11–25. What is the difference in tension between the two sides of a belt running 3000 ft a minute and transmitting 250 hp?

11–26. Find the speed of a driving pulley $2\frac{1}{4}$ ft diameter to transmit 10 hp, the driving force of the belt being 120 lb.

11–27. The average width of an indicator diagram for one side of the piston is 1.6 in.; for the other 1.5 in. Each inch of width represents a steam pressure of 36 lb per sq in. The piston is 10 in. diameter, stroke 1 ft, revolutions per minute 120. What is the indicated horsepower if we assume both sides of the piston of equal area?

SUMMARY

Work, in its technical sense, is the product of force times the displacement in the direction of the force. Force alone does not constitute work. Work is ordinarily measured in foot-pounds.

$$W = Fs \tag{28}$$

Power is the rate of doing work. It equals work divided by time.

$$P = \frac{W}{t} = \frac{Fs}{t} \tag{29}$$

Power is ordinarily measured in horsepower, equivalent to work done at the rate of 33,000 ft-lb per min.

$$\text{Horsepower} = \frac{\text{foot-pounds per minute}}{33{,}000} \tag{29a}$$

Electrical power is measured in watts, or in kilowatts (= 1000 watts).

$$\text{Watts} = \text{volts} \times \text{amperes} \tag{30}$$

$$1 \text{ hp} = 746 \text{ watts} \tag{30a}$$

Belt horsepower can be found by the equation:

$$\text{Horsepower} = \frac{(T_1 - T_2)\, v}{33{,}000} \tag{31}$$

also:

$$\text{Horsepower} = \frac{F \pi D n}{33{,}000} \tag{32}$$

When torque is given:

$$\text{Horsepower} = \frac{2 \pi n T}{33{,}000} \tag{33}$$

For a steam engine:

$$\text{Horsepower} = \frac{pLAn}{33{,}000} \tag{34}$$

The student should assure himself that he knows the principles underlying the foregoing equations. Numbers (31) to (34) are simply special cases of the basic relationship of (29*a*).

CHAPTER XII

ENERGY AND MOMENTUM

The preceding chapter treated of work and power, in its technical sense of force times displacement. In this chapter consideration is given to work as energy stored up or released. Work is done in lifting the hammer of a pile driver; likewise work is done by the falling of the hammer. Between the two operations there is a period in which the work, or energy, is stored up, ready to be released. These topics are treated in Chapter XII. Other topics include the conservation of energy, the transformation of energy, and the meaning and application of the term momentum. The important application of kinetic energy in rotating machinery is also treated.

87. Energy. Energy is the capacity for doing work. A body which is capable of doing work is said to possess **energy.**

Thus, we say that a clock spring when coiled possesses energy because, in unwinding, it can do work in driving the clock mechanism; that a moving projectile possesses energy because it can overcome the resistance offered by the air, by armor plate, etc., and thus do work. A storage-battery possesses energy because it can furnish an electric current to operate a motor, etc.

A body possesses energy as a result of *the fact that at some previous time work has been done upon it.* Thus in the illustrations just given, the coiled spring received its energy when work was done on it in winding it up; the moving projectile acquired the energy it did not have while lying at rest in the cannon when the gases produced in the explosion of the powder, by their pressure, did work upon the projectile, thus overcoming the resistance offered by its inertia and imparting motion; the storage-battery received its energy when " charged " from the electric power line.

Work is therefore *the act of transferring energy.* The body which is doing the work gives or expends energy, the body worked upon receives energy. The energy thus transferred may be transformed into heat during the process, and thus be no longer available for doing work; or a part of it may be stored up as available energy in the body worked upon, as when this body is raised to a higher

level, when it is put in motion, etc. When energy is thus stored, the body worked upon becomes in turn capable of doing work.

The amount of available energy possessed by a body is measured by the amount of work the body can do and is therefore expressed in the same units as work.

88. Potential energy. Mechanical energy is of two kinds. **Potential energy is energy of position.** Thus, if a compressed spring in extending can exert a force of 60 lb over a distance of 4 in., it has when compressed a *potential energy* of 240 in-lb; or, if a body weighing 10 lb is suspended so that when released it may freely fall under the action of gravity 5 ft, it has a *potential energy* due to its position of 50 ft-lb.

$$\text{Potential energy} = W \times h \tag{35}$$

Example.

What is the potential energy of a pile-driver hammer weighing 1 ton when raised a distance of 15 ft?

Solution.

$$\text{Potential energy} = Wh = 2000 \times 15 = 30{,}000 \text{ ft-lb} \quad \textit{Ans.}$$

89. Kinetic energy. Kinetic energy is energy due to motion. A body in motion can be brought to rest only through a resistance opposing the motion. The product of this resistance by the distance through which it acts in stopping the body measures the work stored in the body, and therefore the kinetic energy which it possesses. Thus the kinetic energy of a moving automobile is measured by the work it can do while coming to rest after the clutch has been disengaged, which is equal to the resistance due to the friction, brakes, etc., used in stopping the automobile, multiplied by the distance moved while it is coming to rest.

The kinetic energy of a body may be computed from its mass and velocity as follows:

Kinetic energy = work a body can do because of its motion = work necessary to stop its motion = retarding force $\times$ distance through which it acts in stopping the body. Therefore, in symbols, letting F = the retarding force,

$$\text{Kinetic energy} = Fs$$

But $$F = ma$$

Therefore, $$\text{Kinetic energy} = Fs = mas$$

From equation (14R) $v = \sqrt{2as}$ and $as = v^2/2$.

Therefore substituting $v^2/2$ for as gives:

$$\text{Kinetic energy} = \frac{mv^2}{2} \tag{36}$$

in which, as before, $m = \frac{W}{g}$.

Example.

Find the kinetic energy of a hammer blow in driving a nail, if the hammer weighs 5 lb and has a velocity of 40 fps.

Solution.

$$\text{Kinetic energy} = \frac{mv^2}{2} = \frac{Wv^2}{2g} = \frac{5 \times 40 \times 40}{2 \times 32} = 125 \text{ ft-lb} \quad Ans.$$

90. Transformation of energy. *Potential energy* may be transformed into an equal amount of *kinetic energy*, and vice versa, the sum of the two kinds of energy always remaining constant unless some of it is used up in overcoming friction, or in doing other work. Thus, if a body has potential energy, due to its position, and is allowed to fall, its potential energy decreases as it descends. But its kinetic energy due to its velocity increases at the same rate, until the body has, just as it is reaching the end of its fall, no potential energy, but an amount of kinetic energy equal to the potential energy which it had at the start. If a body weighing W lb is suspended h ft above the ground, its potential energy is Wh ft-lb. If it falls freely, its kinetic energy on striking the ground $= Wv^2/2g = Wh$ ft-lb.

Similarly, if a body has a certain amount of kinetic energy, due to its velocity, and if this velocity is directed upward and we may neglect frictional resistance, the velocity of the body, and therefore its kinetic energy, will decrease as it rises; but its *potential energy* will increase in the same ratio as it rises until, when its velocity has been reduced to zero and it is just starting to fall, its potential energy will exactly equal the kinetic energy which the body had in the beginning.

In addition to what has here been termed " mechanical energy," energy may exist in other forms, as heat (a form of molecular kinetic energy), electrical energy, chemical energy, etc. Under proper conditions, energy of one form may be transformed into another form. Thus, in machinery, the energy expended to overcome friction in the bearings is transformed into heat. In im-

properly constructed or poorly lubricated bearings the amount of energy thus transformed, i.e., the "wasted work," may be considerable, and parts of the machine may be injured from undue heating.

Or, heat energy may be transformed into mechanical energy, as in the steam engine. Whenever heat energy is transformed into mechanical energy, or mechanical energy into heat, it has been definitely determined that the transformation always occurs in accordance with the law that:

The heat energy required to heat 1 lb of water 1° F, called a Btu (British thermal unit), = 778 ft-lb. (37)

Mechanical energy is transformed into electrical energy in a generator, and electrical energy into mechanical in a motor. The relation between the mechanical and electrical units is the same as for power, Art. 84.

Electrical energy is transformed into heat in electric heaters, lights, toaster stoves, etc. *The relation between heat and electrical units is:*

$$1 \text{ Btu} = 1055 \text{ watt-seconds}$$

and **1 watt-hour = 3.4 Btu** (38)

91. Law of the conservation of energy. This law, briefly stated, is:

1. Energy cannot be created or destroyed. The total sum of energy in the universe remains constant.
2. Energy may be transformed from one form to another; in such transformation no energy is ever destroyed.

These propositions have been established by observations and experiments extending over many years, in all branches of natural science, especially in physics and chemistry. They have been generally accepted, and form the basis of many theories not only in engineering but also in biological and physical science. The principle of the conservation of energy thus becomes both an aid to the understanding of many natural processes and a test by which science determines the reasonableness of a new theory or hypothesis.

The definite statement of the law of the conservation of energy dates from the middle of the nineteenth century, when it was experimentally shown that heat is a form of energy, and that heat may

be transformed into mechanical energy or mechanical energy into heat. It was also determined experimentally, that where such transformation from one form of energy to another form occurs, there is always the following quantitative relationship:

The heat required to raise the temperature of 1 lb of water 1° F. is equivalent to 778 ft-lb of mechanical energy.

The equivalence of various forms of energy may be stated in many other cases, which include such apparently different ideas as electrical and chemical relations and the phenomena of sound, light, and animal and vegetable processes. Thus take 1000 ft-lb of mechanical energy and by proper mechanism, either natural or of human devising, transform it into other forms of energy; we find that always a certain perfectly definite amount of chemical or electrical or other energy is obtained and that by no device can we ever produce more energy than the 1000 ft-lb we started with. It is perhaps unnecessary to add that the complete tracing of these transformations and the measurement of the amounts at all stages of the processes is not humanly possible, since the processes are as intricate and manifold as nature itself. All our experiments, however, point undeniably to the truth of the great fact that energy may be transformed but never destroyed.

It must not be inferred that the law of conservation of energy means that the supply of energy *available for our use* is constant. In all machinery by which energy is utilized, there is a constant loss of useful energy through the transformation of a portion into heat

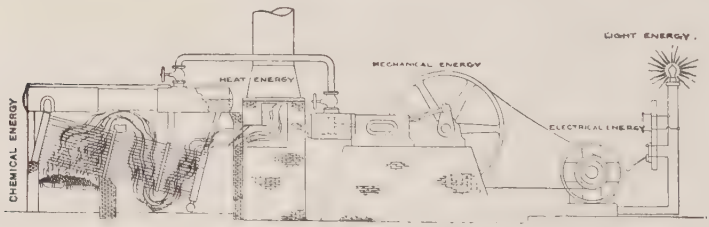


FIG. 182. The transformation of energy.

energy by friction at bearings, etc., and the dissipation of this heat. Thus, as an illustration, suppose 10,000 units of heat energy are liberated by the combustion of 1 lb of coal in the boiler furnace. Of this about 7500 units are given to the steam, the other 2500 units are dissipated through the ash, chimney gases, radiation, etc.

Of the 7500 units carried into the cylinder about 800 units are transformed into mechanical energy of the moving piston, 6700 units being lost through condensation, radiation, etc., in the cylinder, or carried on into the condenser. If the engine is driving a dynamo, about 700 of the 800 units may reappear in the energy of

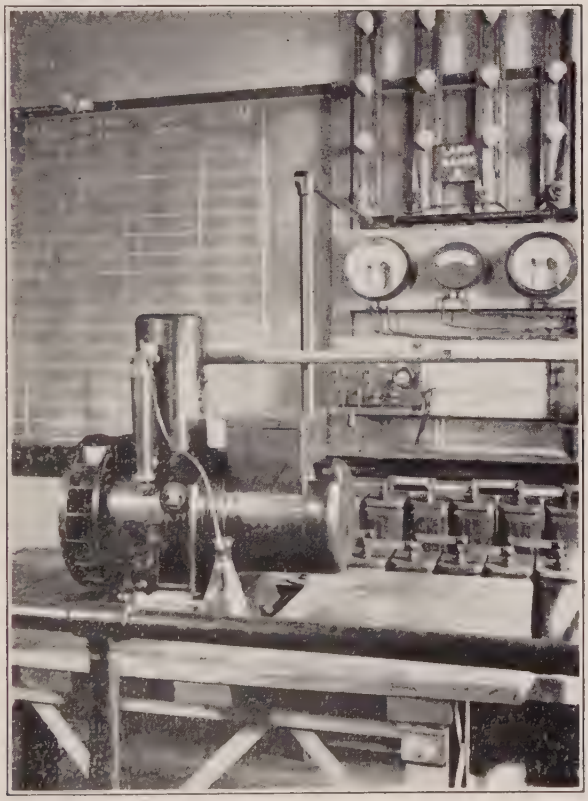


FIG. 183. 1-hp. light and power unit. Energy from kerosene in the flask is transformed through several stages into light.

the electric current, the other 100 disappearing as heat through friction, electrical losses, etc. If the dynamo is supplying arc or incandescent lights, the 700 units are given to heat the connecting wires, lamp filaments, etc., and thus all our original supply of energy has been dissipated. Or, if the dynamo drives a motor, about 650 units may be obtained in mechanical energy supplied

by the motor, 50 being lost in the transformation through friction, electrical losses, etc. Of the total 10,000 units with which we started, therefore, approximately 93.5 per cent have been dissipated, 6.5 per cent are still available for doing work.

Many similar examples might be given. In all we find that each step in the process of utilizing energy is accompanied by unavoidable loss.

PROBLEMS

12–1. What is the potential energy of a body weighing 2000 lb at a height of 80 ft above the ground? What will be its velocity on reaching the ground if allowed to fall freely, and what will be its kinetic energy on striking?

12–2. A 5-ton pile-driver falls freely 6 ft. Assuming all of its kinetic energy to be expended in sinking the pile, against what average resistance can it drive the pile 1 in.?

12–3. A reservoir contains 50,000 gal water at an average elevation of 200 ft above a given level. What is the potential energy of the water with respect to this level?

12–4. If the mass of a moving body be doubled and its velocity at the same time be increased fourfold, how is the kinetic energy changed?

12–5. A car weighing 10 tons is moving at the rate of 36 mph. Find the average force required to stop it within a space of 50 ft.

12–6. A bicyclist weighing 155 lb mounted on a wheel weighing 25 lb, rides at a velocity of 25 mph on a horizontal plane. If he comes to a slope rising 12° and removes his feet from pedals, how far up the slope will he go?

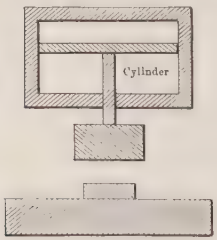


Fig. 184.

12–7. The steam hammer shown in the diagram of Fig. 184 has a cylinder 72 sq in. in cross section, the stroke is 20 in., the steam pressure is 80 lb per sq in., the weight of the hammer, including piston and piston-rod, is 644 lb. Find the energy which the hammer will have after one full stroke. How far will the hammer blow sink into a piece of copper if the average resistance offered by the copper is 50 tons?

Note. Work done by gravity and work done by steam, *both* give velocity to hammer. Disregard friction.

12–8. The section of a stream is 10 sq ft and the average velocity of the water 2 fps; there is an available fall of 30 ft. What is the available supply of energy per minute?

12–9. A ball weighing $\frac{1}{4}$ lb strikes a shield with a velocity of 1000 fps. It pierces the shield and moves on with a velocity of 400 fps. How much energy was lost in piercing the shield? What average resistance was offered by the shield if 1 in. thick?

12–10. A fire-engine pump is provided with a nozzle 1 sq in. area of section and water is projected through the nozzle with an average velocity of 140 fps. Find:

(*a*) The kinetic energy of each pound of water as it leaves the nozzle.

(*b*) The horsepower of the engine required to drive the pump which supplies the nozzle, assuming that 75 per cent of the energy supplied to the pump is transferred to the water.

12–11. A rapid-fire gun discharges 5 projectiles per minute each of a weight of 100 lb and a velocity of 2000 fps. What is the horsepower expended?

12–12. A hammer weighing $1\frac{1}{2}$ lb has a velocity of 18 fps when it strikes a nail. What is the average force exerted on the nail if it is driven into the wood $\frac{1}{2}$ in.?

92. Energy of rotating bodies. As shown in Art. 89, the kinetic energy of a body moving in a straight line is expressed by half the product of its mass by the square of its velocity. In such motion all portions of the body move in the same direction and with the same velocity. In a rotating body, as, for example, a flywheel, the kinetic energy of any *particular particle* of the wheel is obviously $\frac{1}{2}mv^2$, where m is the mass of the particle, v its velocity, and the kinetic energy of the whole wheel is the sum of the kinetic energy of its particles, or $\sum(\frac{1}{2}mv^2)$, where v has different values according to the distance of the particles from the axis of rotation.

Since in rotary motion the whole mass of the moving body may be assumed as concentrated at a distance equal to the radius of gyration from the axis, a convenient equivalent expression for the kinetic energy of a rotating body is

$$\text{Kinetic energy} = \tfrac{1}{2}mv_k^2 \tag{39}$$

where v_k equals the linear velocity at the distance of the radius of gyration from the axis.

Since moment of inertia, Art. 77, bears the same relation to rotary motion as mass does to linear motion, the kinetic energy of a rotating body may also be expressed as one-half the product of its moment of inertia by the square of its angular velocity, or

$$\text{Kinetic energy} = \tfrac{1}{2}I\omega^2 \tag{40}$$

Example 1.

What is the kinetic energy of a flywheel weighing 3000 lb when running 180 rpm, its dimensions being as follows: outside diameter 6 ft 6 in.; thickness of rim 6 in.?

Solution. The radius of gyration for flywheels, where the spokes and hub may be neglected, is approximately equal to the mean radius of the rim. Hence the kinetic energy in this case will be approximately as if all the mass were placed 3 ft from the center. Therefore,

$$v_k = \frac{6 \times \pi \times 180}{60} = 56.5 \text{ fps}$$

Kinetic energy =

$$\frac{1}{2} m v_k^2 = \frac{W v_k^2}{2g} = \frac{3000 \times 56.5 \times 56.5}{2 \times 32} = 149{,}600 \text{ ft-lb} \quad \textit{Ans.}$$

Example 2.

A flywheel weighing 5400 lb, mean radius 2 ft 9 in. running at 120 turns per minute, gives up 8000 ft-lb of energy; how much is its velocity diminished?

Let v_1 be the velocity in the beginning, and
v_2 be the velocity after the 8000 ft-lb have been withdrawn.

$$v_1 = \frac{2 \times 2\frac{3}{4} \times \pi \times 120}{60} = 34.6 \text{ fps}$$

$$\text{Kinetic energy in beginning} = \frac{5400 \times v_1^2}{2 \times 32}$$

$$\text{Kinetic energy at end} = \frac{5400 \times v_2^2}{2 \times 32}$$

Therefore,

$$\frac{5400 \times v_1^2}{2 \times 32} - \frac{5400 \times v_2^2}{2 \times 32} = 8000$$

Substituting the value of v_1 above and solving

$$v_2 = 32.2 \text{ fps} = 115 \text{ rpm} + \quad \textit{Ans.}$$

PROBLEMS

12–13. What is the kinetic energy of every pound 4 ft from the axis on the rim of a flywheel making 120 rpm?

12–14. The rim of a flywheel weighs 15,000 lb and its mean linear velocity is 40 fps. How many foot-pounds of work are stored in it? If it is required to give out 20,000 ft-lb, how much will its velocity be decreased?

12–15. The flywheel of a gas engine has the following dimensions:

Mean diameter of flywheel, to center of rim, 6 ft
Width of rim, 8 in.
Thickness of rim, 6 in.
Weight of rim per cubic foot, 465 lb

What is the kinetic energy of the wheel when running at 150 rpm?

12–16. How much energy must be supplied to the wheel in Problem 12–15 to increase its speed to 180 rpm?

12–17. A flywheel has a mean radius of 3 ft 4 in. and a normal speed of 122 rpm. It is required to supply 3600 ft-lb from its store of energy while slowing down to 120 rpm. What weight of rim is required?

12–18. A flywheel weighing 160 lb possesses 800 ft-lb of kinetic energy. What is the velocity of the point where all its mass may be regarded as collected?

93. Momentum. Momentum, or quantity of motion, furnishes an additional means of relating force and mass, by considering the speed of a body. A small car running at high speed can do more damage if out of control than a larger one at very slow speed, because of its momentum.

The product of the mass of a moving body by its velocity is called its *momentum*. Or,

$$\text{Momentum} = mv \tag{41}$$

Thus a body of 10 units mass when it is moving with velocity 8 units per sec, has a momentum of 80; a body whose mass is 20 units has also a momentum of 80 if its velocity is 4 units per sec.

The starting and stopping, or *any* change in the velocity of a moving body is governed by the relation $F = ma$. If v = total change in velocity, $a = v/t$. Hence

$$F = \frac{mv}{t} \tag{42}$$

The force acting may be measured by the momentum it can generate or destroy in unit time.

Two methods may therefore be used in computing the force acting in producing a change of velocity. Thus, suppose we are given the *weight* and *velocity* of the body and *the distance* through which the force acts. We may then compute the kinetic energy of the body and apply the principle of work to find the force.

Example 1.

Suppose a hammer weighing 5 lb, moving with a velocity of 40 fps when it strikes a nail, drives the nail 1 in. into the wood.

Solution. The kinetic energy of the hammer is

$$\text{Kinetic energy} = \frac{1}{2}mv^2 = \frac{Wv^2}{2g} = \frac{5 \times 40 \times 40}{2 \times 32} = 125 \text{ ft-lb} \quad \textit{Ans.}$$

This energy is expended in doing work against the resistance R offered by the wood, or $125 = R \times 1\frac{1}{2}$, hence,

$$R = 1500 \text{ lb} \quad Ans.$$

It should be noted that R is here the *average resistance* offered by the wood, over the distance of 1 in. R is, then, the *space average* of the force. The resistance offered by the wood may or may not be uniform; it is equivalent to the uniform resistance of 1500 lb through the whole distance, 1 in.

Or, if we know the *weight* and *velocity* of the body and the *time* during which the force acts, we may compute the momentum of the body and then find the force from the relation $F = mv/t = Wv/gt$.

Example 2.

Thus, suppose the hammer of the preceding illustration is brought to rest in $\frac{1}{200}$ sec.

Solution. The momentum of hammer $= mv = \frac{5}{32} \times 40$. And the resistance R offered by the wood $= \dfrac{Wv}{gt} = \dfrac{5 \times 40}{32 \times \frac{1}{200}} = 1250$ lb *Ans.*

R is here the *time-average* force, i.e., the average force which will stop the hammer in the given time, $\frac{1}{200}$ sec.

This method is useful when considering impact, i.e., impulsive forces or blows, which act for a very short interval of time. The following examples will illustrate the general applications:

Example 3.

A stream of water 2 in. in diameter impinges at an angle of 90° upon a brick wall. If the velocity of the stream is 200 fps, what is the force exerted on the wall?

Solution. The weight of water reaching the wall *per second* is represented by a cylinder of water 2 in. in diameter and 200 ft long, or $\frac{1}{12} \times 1\frac{1}{2} \times \pi \times 200 \times 62.5 = 273$ lb. The change of velocity is equal and *opposite* to the velocity of the stream, or 200 fps. Therefore reaction of wall

$$= \frac{273 \times 200}{32 \times 1} = 1700 \text{ lb} \quad Ans.$$

Example 4.

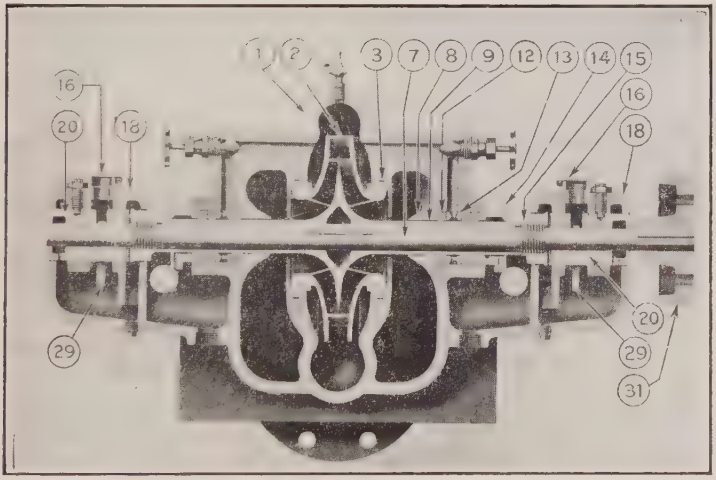
120 cu ft of water leave the rim of the wheel of a centrifugal pump every minute. The component velocity of the water in the direction of the motion of the rim is 20 fps, and the velocity of the rim is 25 fps. If the water enters at the center with no velocity in the direction of the rim, what horsepower will be required to drive the pump, neglecting friction?

Solution. 120 cu ft per min = 2 cu ft each sec = $2 \times 62.5 = 125$ lb of water each sec. To impart to this a velocity of 20 fps requires a force of

$$\frac{125 \times 20}{32 \times 1} = 78.1 \text{ lb}$$

If the velocity of the rim is 25 fps, the work done against this reaction of 78.1 lb at the rim of the wheel is $78.1 \times 25 = 1940$ ft-lb per sec. The horsepower required is therefore

$$\frac{1940 \times 60}{33{,}000} = 3.55 \text{ hp} \quad \textit{Ans.}$$



Courtesy Allis-Chalmer's Co.

FIG. 185. Centrifugal pump. 2 is the runner, 7 the shaft.

94. The conservation of momentum. Since a force is measured by the momentum it will generate in unit time, it follows that *equal forces, acting for the same time, will generate equal momenta.* The momentum imparted to a gun when fired, if the gun can be considered as free from other bodies, will equal the momentum imparted in the opposite direction to the bullet, although because of its greater mass the velocity of the gun will be very small in comparison. In the case of colliding bodies, as when a body A strikes another body B, the mutual pressures between the bodies are equal and opposite during the time of impact. If A loses momentum, B must gain an equal amount provided the action of

outside bodies upon both can be neglected. The total momentum of the system (i.e., of A and B taken together) therefore remains the same as it was before impact, although the velocities of both A and B may be changed in either amount or direction or in both. This fact that *the total momentum of a system remains constant unless a force external to the system is applied* is known as **the law of conservation of momentum.**

It must not be inferred from what has been said, however, that the total *mechanical energy* of the system remains constant also. During impact, work is done against internal forces (cohesion, internal friction, etc.), and some mechanical energy is thus transformed into heat energy. The mechanical energy is therefore always less after impact.

PROBLEMS

12–19. A truck weighing 12 tons, moving with a velocity of 5 fps, is stopped by buffers in $\frac{1}{3}$ sec. What is the average force of the blow?

12–20. A ship of 10,000 tons moving at 5 mph is stopped in one minute. What average force is required?

12–21. A gun delivers 100 bullets per min, each weighing one ounce, with a horizontal velocity of 1610 fps. What is the average force exerted upon the gun?

12–22. A $1\frac{1}{2}$-in. stream from a fire engine is thrown horizontally against a wall with a velocity of 150 fps. What force is exerted against the wall?

12–23. A centrifugal pump discharges 80 cu ft of water per min at a velocity tangent to the rim of the wheel of 25 fps. If the velocity of the rim is 30 fps, what is the horsepower required?

SUMMARY

Energy is the capacity for doing work. **Potential energy** is energy of position, or stored energy. **Kinetic energy** is energy due to motion.

$$\text{Potential energy} = Wh \tag{35}$$

$$\text{Kinetic energy} = \frac{mv^2}{2} = \frac{Wv^2}{2g} \tag{36}$$

Energy may be **transformed** from potential into kinetic, and vice versa; also from the mechanical form into heat or electrical or chemical energy, and vice versa. The relations of the various units are:

$$1 \text{ hp} = 746 \text{ watts} \tag{30a}$$

$$1 \text{ Btu} = 778 \text{ ft-lb} \tag{37}$$

$$1 \text{ watt-hour} = 3.4 \text{ Btu} \tag{38}$$

The relations for energy of rotation are:

$$\text{Kinetic energy} = \tfrac{1}{2} mv_k^2 \quad \text{also} \quad = \tfrac{1}{2} I\omega^2 \qquad (39), (40)$$

Momentum is called the quantity of motion, and equals the product of mass times velocity. Momentum gives an additional method of analyzing the relations of force and mass in moving bodies.

$$\text{Momentum} = mv \qquad (41)$$

$$F = \frac{mv}{t} \qquad (42)$$

The law of the conservation of energy states that energy cannot be created or destroyed, although it may be transformed from one form into another.

The law of the conservation of momentum is that the total momentum of a system remains constant, unless a force external to the system is applied.

CHAPTER XIII

FRICTION

The theoretical principles of mechanics must often be modified in practice because of the fact that friction, due to moving parts, introduces supplementary forces. These forces are variable in amount and in direction so that the best that can be done is to make an assumption, based largely upon experimental data. Various textbooks and handbooks give tables and approximate formulas to cover the cases commonly encountered. In this chapter only the fundamental ideas will be considered, and the student is referred to engineering textbooks for more complete information, particularly in connection with friction of bearings and belts, rolling and tractional friction, etc. Friction of machines will be treated in Chapter XIV; friction of fluids and friction in pipes and airducts in Chapter XVI.

95. Friction. Friction is the resistance which a body offers to the sliding of another body across its surface. It takes a force to slide a block along a table top or to pull a sled along the ground. This force, if the motion is uniform, is the measure of the friction between the two sliding surfaces. It requires a greater force to start the body sliding than to keep it moving uniformly when once started, that is, starting or *static* friction is greater than *sliding* or *kinetic* friction.

Friction is sometimes an advantage and sometimes a disadvantage. It is an advantage when it gives stability against the sliding of walls and foundations; also in preventing belt slip on pulleys, in holding machines such as a screw jack in position under a load, etc. It is a disadvantage in moving machine parts, such as shaft bearings; and in the flow of water through pipes, etc. Friction is absolutely necessary in automobile clutches and in car brakes. In brake bands and bearings sliding friction is encountered; whereas in belting on pulleys and in building joints it is static friction.

Certain facts or "laws of friction" have been developed, and these differ for dry and for lubricated surfaces. These laws are necessarily approximate rather than exact, but they give a

standard to guide the designer. Friction between dry surfaces varies with the material and the nature and condition of the surface. It is proportional to the pressure between the surfaces. It is independent of the area of contact. It varies only slightly with the speed, being greater at slow speed.

Suppose a cast-iron block weighing W pounds rests upon the horizontal surface AB, also of cast iron, as in Fig. 186. The pressure between the surfaces in contact will be W pounds. A force F_1 will be required to cause the body *W to start from rest; a smaller force,* F_2, *will keep the body moving uniformly when once started.* F_1, therefore, measures the force of starting or *static friction* and F_2 measures the force of friction of motion or *sliding friction.* If the pressure W between the surfaces be increased by adding weights to the block, a larger force will be required to start the block than was needed before, and also a larger steady pull to keep it sliding steadily. In all cases, however, F_1 will be greater than F_2, i.e., *static friction is greater than sliding friction.*

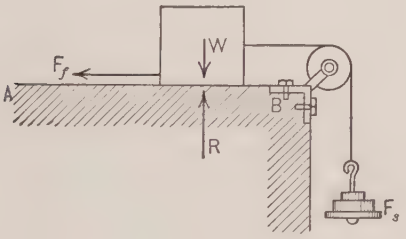


Fig. 186.

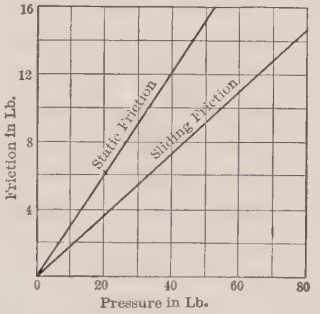


Fig. 187. Relation of friction to pressure between the surfaces.

If a series of determinations of both sliding and static friction be made for different pressures between the surfaces in contact, the results can be shown graphically, as in Fig. 187. The resulting curves will be straight lines passing through the origin. They show, therefore, that for both static and sliding friction, the *force of friction is proportional to the pressure perpendicular to the surfaces in contact,* and therefore that:

$$\frac{\text{Friction}}{\text{Pressure}} = \textbf{a constant value,} \text{ for the same surfaces}$$

If the area of the surfaces in contact is changed, keeping the total pressure between them unchanged, as, for example, if W is placed upon its narrow side, approximately the same values for the force of friction will be obtained provided this side has been dressed in the same manner as the other. The force of *friction is therefore practically independent of the area of the surfaces in contact*, where cutting does not occur.

If instead of cast-iron surfaces we use brass on cast iron; cast iron on leather; oak on oak, fibres parallel; oak on oak, across the grain, etc.; we will still obtain curves which are straight lines passing through the origin, but each curve will have a different slope. In other words, the fraction $\frac{friction}{pressure}$ will be a constant for each case, but will have different values for each case.

96. Coefficient of friction. The coefficient of friction for any two surfaces is the figure obtained by dividing the friction by the total normal pressure, that is, the force between the surfaces. For horizontal surfaces the normal pressure is the weight. Letting F represent the force of friction, and f the coefficient of friction, $f = F/W$. For surfaces which are not horizontal, the normal pressure will not equal the weight but will be somewhat less. Thus on the inclined plane in Fig. 158, call the normal component of the weight $N = OC = W \cos \delta$. The force required to move a body up the incline must overcome the gravity component parallel to the plane and the friction F. For this condition the relation is:

$$f = \frac{F}{N} \tag{43}$$

which is the general equation, since for a horizontal surface, $N = W$.

The symbol f is the coefficient of static or of sliding friction, dependent upon whether F is the force to overcome starting or sliding frictional resistance. When desired to distinguish the force of friction from some other force in the same computation the symbol F_f may be used for the force of friction. To distinguish between starting and sliding (kinetic) friction the symbols may be F_s for starting or static friction, and F_k for kinetic friction. Then f_s and f_k will represent the coefficients of static and sliding friction respectively.

The values of f are listed in reference books, so that the engineer may determine the probable resistance to sliding between two surfaces.

Transposing equation (43) gives $F = fN$, that is:

The force required to slide one body on another = the coefficient of friction × the normal pressure between the surfaces.

For example, suppose that the force to just *start* a brick sliding along a concrete foundation is 3 lb and that the brick weighs 5 lb. Whether the brick lies flat or on edge, the friction should be about the same if the texture is uniform. If the brick weighed twice as much the frictional force would be twice as much. In either case $f = F/N = \frac{3}{5} = 0.60$ for brick on concrete. Suppose further that a section of brick wall (of the same texture brick) weighs 1200 lb. The side pressure (as of earth embankment) that it could resist before it would start sliding $= F = f \times N = 0.60 \times 1200 = 720$ lb.

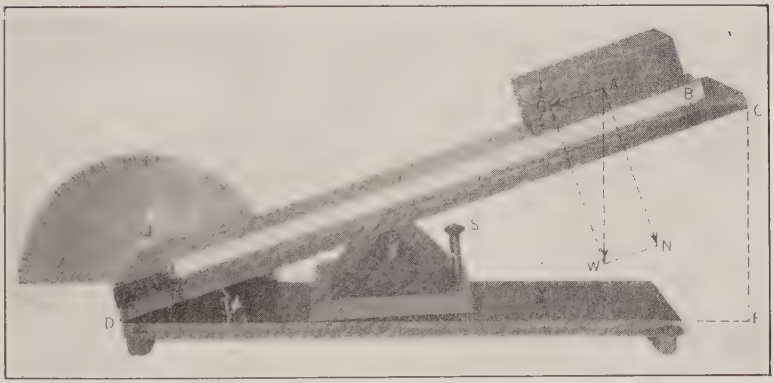


FIG. 188. Inclined plane for determining the coefficient of friction.

97. Determination of the coefficient of friction. The coefficient of friction between two surfaces may be obtained by determining the force F which will just keep a body moving at a uniform rate when once started, as in Fig. 186. A more convenient form of apparatus is shown in Fig. 188. CD is an inclined plane which can be adjusted to any desired angle of slope. Upon this are placed the surfaces to be tested, A and B, A being free to move. The weight, W, of A may be resolved into the two forces G, tending to

cause A to slide on B, and N, perpendicular to the line of contact between A and B, and producing pressure between them.

Rough adjustment of CD is made by moving the triangular block T forward or back. Fine adjustment is then made by tipping the block by turning the screw S until, on tapping A to start it, it slides uniformly on B.

Thus friction $= G$, and $f = G\ N = CE\ DE$ = tangent of the angle of slope, CDE.

The angle at which friction F just equals gravity component G is called the *angle of friction* and is represented by ϕ. Then the coefficient of friction equals the tangent of this angle, that is

$$f = \tan \phi \tag{44}$$

98. Rolling Friction. Rolling friction is less than sliding friction because the rolling motion lifts any projections on the rolling body out of the depressions in the surface rolled upon, whereas in sliding friction the projections on both surfaces must be broken off. It may be shown experimentally that rolling friction is approximately proportional to the pressure between the surfaces and inversely proportional to the radius of the rolling body.

Rolling friction is greatly dependent upon the nature of the surface rolled upon. Where this surface is soft so that the wheel or roller makes a depression, the condition shown in Fig. 189 results. The surfaces are in contact not at a line only, but over a considerable area. There is slipping at the surfaces in contact as well as rolling. Energy is expended not only in deforming the surface but also in constantly climbing the ridge formed. When a vehicle on wheels is being moved, the **tractional resistance** is the sum of all the resistances opposing motion, wherever located. Tractional resistance, therefore, includes friction at the ground, at the axles of the wheels, and friction of the air. Some of these, notably the frictional resistance due to the air, depend upon the speed. For the same vehicle, at ordinary speeds, the law $F = Nk$ applies, where F is tractional resistance, N is normal pressure, and k

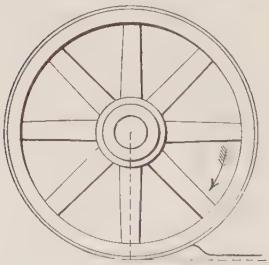


Fig. 189. Conditions affecting rolling friction.

is an abstract number called the **coefficient of traction.** The approximate value of k is 0.10 for a dirt road, 0.04 for a macadam road, 0.015 for an asphalt pavement, and 0.003 at slow speed for a railroad track.

99. Laws of friction. Fluid friction. The laws for the friction of fluids differ markedly from those for dry surfaces. We have noted that the factor having greatest effect on the friction of dry surfaces is the pressure. When the pressure becomes too great, the surfaces may "seize." The friction of dry surfaces, while not greatly changed, is less for high speeds. On the contrary, the friction of fluids is independent of the pressure, and increases rapidly as the speed is increased. Beyond a certain "critical speed," depending on the liquid used and its temperature, fluid friction is proportional to some power of the speed. The frictional resistance for high-speed motor boats and airplanes is very great. The resistance to the motion of a rifle bullet may increase with the velocity until proportional to the square root of the fifth power of the velocity; in other words, a bullet at four times the velocity may meet with thirty-two times the frictional resistance from the atmosphere.

The laws of fluid friction, and comparisons with the corresponding laws for the friction of dry surfaces, are given in the following summary from Perry's *Applied Mechanics*:

LAWS OF FRICTION

Friction of Dry Surfaces	Fluid Friction
1. The force of friction does not much depend on velocity but is greatest at slow speeds.	1. The force of friction very much depends on the velocity, and is indefinitely small when the speed is very slow.
2. The force of friction is proportional to the total pressure between the two surfaces.	2. The force of friction does not depend on the pressure.
3. The force of friction is independent of the areas of the rubbing surfaces.	3. The force of friction is proportional to the area of the wetted surfaces.
4. The force of friction depends very much on the nature of the rubbing surfaces, their roughness, etc.	4. The force of friction at moderate speeds does not much depend on the nature of the wetted surfaces.

100. Friction of lubricated surfaces. The differences between the friction of dry bodies and of fluids, as stated in the preceding article, should be noted. Even with dry surfaces the amount of air film between the surfaces may greatly modify the friction. Lubricated surfaces, as ordinary bearings, etc., would naturally obey neither the laws for dry surfaces nor those for fluids, but would be intermediate between the two, varying from the case of flooded bearings, which follow quite nearly the laws for fluids, to very modified and uncertain conditions as the degree of lubrication, kind of lubricant, temperature, pressure between surfaces, velocity, etc., change. No very definite statements can therefore be made for the friction of lubricated surfaces.

In general, the character of the lubricant should be such that it will be constantly drawn between the rubbing surfaces as it is needed. Where the pressure between the surfaces is great, light, thin oils would be forced out, hence for such cases heavy oils or grease are preferable. Under light pressures these would be too viscous.

The properties of oils vary greatly with temperature also, and the selection of a lubricant must depend considerably upon the temperature to which it will be subjected. For high temperatures, such as result when high-pressure steam is used, mineral oils which do not decompose with heat must be used.

101. The friction of bearings. The horsepower expended in driving a shaft, since $F = fN$, is:

Horsepower = the coef. of friction at the given speed $\times$ the pressure of the shaft on the bearings $\times \pi \times$ the diameter of the shaft $\times$ revolutions per minute $\div$ 33,000.

Poor alignment greatly increases the pressure of a shaft on its bearings and therefore the friction. Any computation based upon the preceding equation, however, is necessarily only approximate owing to lack of knowledge of the friction at a particular speed. Figures 190(*b*) and (*c*) show the results of tests of the horsepower required to keep a shaft turning at constant speed in bronze, cast iron, and roller bearings at 130 rpm and 460 rpm with the same loads. At a speed approximately 3.5 times as fast, the horsepower for cast iron was found to be only 2.5 and for roller bearings 2.9 times as much; or, in other words, in these cases the friction

was less at a high speed. In the bronze bearing, on the contrary, the horsepower given by the test was 5 times as much, showing an apparent increase of friction with the speed.

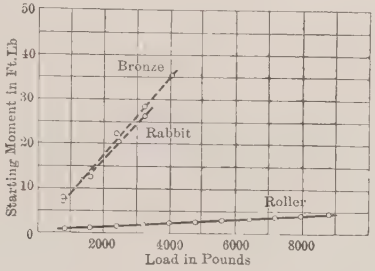


FIG. 190(*a*). Starting torque for different bearings.

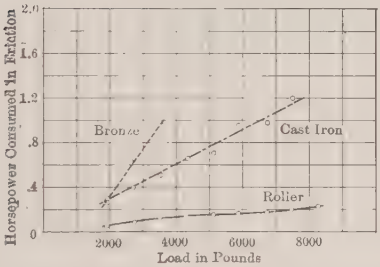


FIG. 190(*b*). Horsepower consumed in friction. Speed 130 revolutions per minute.

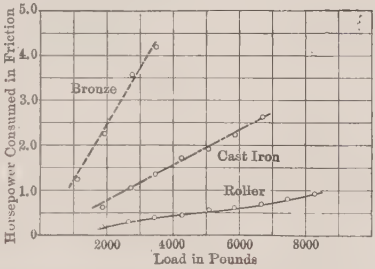


FIG. 190(*c*). Horsepower consumed in friction. Speed 460 revolutions per minute.

Friction at the bearings of a shaft or spindle can be reduced by the use of **ball bearings.** These consist of a number of hard steel

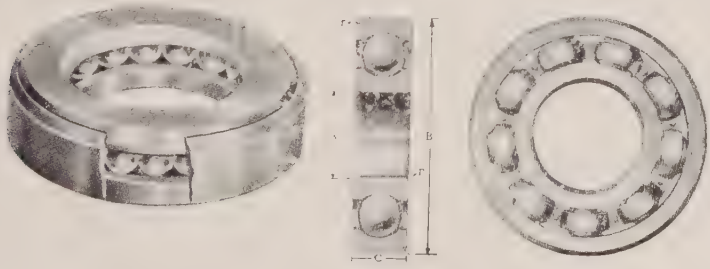


FIG. 191(*a*). Ball thrust-bearing. (*b*). Ball ring-bearing.

balls placed between hardened steel surfaces of proper curvature. The sliding friction which would be encountered if a plain bearing

were used is thus greatly reduced, as the friction of the balls on the bearing surfaces is rolling friction and there is sliding friction only at the points of contact between the balls where the pressure is not great. **Roller bearings** are similarly substituted for plain bearings where the pressure is too great to permit the use of ball bearings. See Figs. 191 and 192.

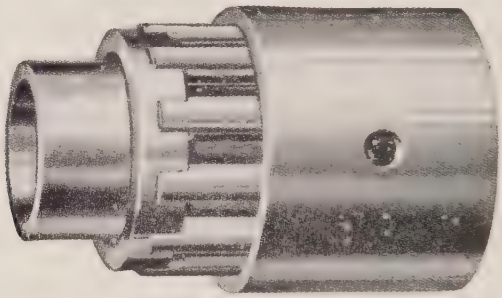


Fig. 192. Roller-bearing.

102. Friction of belts. Suppose a pulley P_1 is driving a second pulley P_2 by means of a belt, the direction of rotation being as shown by the arrows, Fig. 193; both sides of the belt, T_1 and T_2, must be under tension in order to give pressure on the pulleys and therefore friction which shall keep the belt from slipping. But the tension T_1 must be greater than the tension T_2, otherwise P_2 will not turn.

The friction of a belt on the pulley may be increased by increasing the tension — and therefore the pressure between the belt and

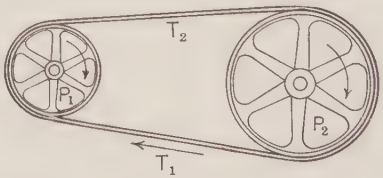


Fig. 193. Belt friction. T_1 is greater than T_2.

pulley — or by increasing the arc of contact between the belt and pulley. The usual practice is to drive with the *upper side* of the belt the slack side, so that any sagging due to the weight of the belt may increase the arc of contact. "Idlers" or tightening pulleys are often placed on the upper or slack side of the belt to increase both the tension and the arc of contact.

PROBLEMS

13–1. If it requires 23 lb to start a 52-lb block of iron sliding on an oak table top, and 19 lb to keep it moving uniformly, what are the coefficients of static and sliding friction?

13–2. If it takes a horizontal force of $37\frac{1}{2}$ lb to pull a box weighing $62\frac{1}{2}$ lb over a cement floor, what is the coefficient of sliding friction?

13–3. If the coefficient of friction for wood on stone is 0.60, how heavy a wooden crate can be pulled over a stone floor by a horizontal force of 80 lb?

13–4. (*a*) If the coefficient of friction for metal on metal, oiled, is 0.05, how much force is lost in a sliding machine part if the weight is 300 lb? (*b*) How much if the surfaces are dry and the coefficient is 0.20?

13–5. If the coefficient of static friction of masonry on masonry is 0.65, what is the stability against sliding of a section of masonry wall, 1 ft in length, 10 ft high, 4 ft wide at the base, and 3 ft wide at the top, if it stands on a masonry footing?

13–6. If the average pressure of a 2-in. brake band on a 14-in. diameter brake drum is 12 lb per sq in., what is the frictional resistance with a coefficient of 0.30?

13–7. A block weighing 200 lb rests on a horizontal surface. It requires 90 lb pull to slide it along with uniform speed. Find the coefficient of friction and the work done in sliding the block 20 ft.

13–8. If it requires a force of 65 lb to move a body weighing 100 lb *uniformly* up a plane inclined 30°, find the coefficient of friction for the surfaces.

13–9. Compute the work done in sliding a wooden body weighing 300 lb up an inclined plane 5 ft high and 20 ft long, the coefficient of friction being 0.4.

13–10. A block of stone is hauled along the ground by a pull of 310 lb at an angle of 10° above the horizontal. If the coefficient of friction is 0.58, what is the weight of the stone?

13–11. A wagon weighing 2200 lb is pulled up a hill rising 2 ft in 20 ft of road. If the tractional resistance is 12 lb per hundred weight, what pull parallel to the road must be exerted?

13–12. What is the *least* pull which will move a box weighing 200 lb along a concrete floor if the coefficient of friction is 0.50?

Note. *Least* pull acts at an angle above the horizontal whose tangent equals coefficient of friction.

13–13. An inclined plane of polished oak is adjusted until a block of cast iron weighing 12 lb will just slide uniformly down. If the plane measures 36 in. long, and is raised 7 in. when adjusted, what is the coefficient of cast iron on polished oak? What force will slide the block uniformly up the plane if it is elevated 5 in. in the 36-in. length?

13–14. If the coefficient of static friction between steel and wood is 0.55, at what height could a board 30 in. long be elevated before a steel block weighing 15 lb would start to slide down?

13–15. If the coefficient of sliding friction of sand on sand is 0.35, at what slope would an embankment of sand remain stable?

13–16. A cast-iron flywheel has a rim 6 ft mean diameter, 7 in. wide, and 4 in. thick. It is keyed to a shaft 4 in. diameter, weighing 500 lb. If the flywheel is running 148 rpm and the coefficient of friction between the shaft and bearings is 0.025, approximately how many revolutions will the flywheel make in coming to rest?

13–17. What horsepower will be required to drive the loaded shaft of Problem 13–16, at the uniform speed of 240 rpm?

13–18. A 10-lb weight on a table is acted on by a horizontal force of 7 lb. The coefficient of sliding friction for the materials is 0.50. Find the acceleration and the distance moved in 2 sec.

13–19. If an automobile weighs 3200 lb and the coefficient of friction on all four wheels is 0.65, what is the braking force available for stopping? What will be the maximum negative acceleration possible?

13–20. If the coefficient of friction of carbon on copper is 0.24 and if the average pressure of the carbon brushes upon a 14-in. diameter commutator of a motor is 2 lb per square inch, and the total area of contact 10 sq in., what will be the horsepower used in overcoming the friction of the brushes at a speed of 600 rpm?

13–21. A N. Y. C. locomotive weighing 200 tons and exerting a tractive force of 20,000 lb hauls a trailing load of 400 tons. If the tractional resistance is $6\frac{1}{2}$ lb per ton, how long will the locomotive require to get up a speed of 55 mph from rest: (*a*) On a level? (*b*) On a 1 per cent grade?

13–22. What is the maximum trailing load that the locomotive of Problem 13–21 can keep moving at a constant speed of 45 mph up a 1 per cent grade?

SUMMARY

Friction is the resistance which one body offers to the sliding of another across its surface. It is measured by the force of friction required to overcome the resistance. The force to just start a body from rest is called static friction; the force to keep it sliding uniformly (horizontally) is called sliding or kinetic friction.

The **coefficient of friction** is the ratio of the friction to the normal pressure, or force, between the surfaces. Expressed in symbols:

$$f = \frac{F}{N} \quad \text{or} \quad = \frac{F_f}{N} \tag{43}$$

To distinguish starting from sliding friction this equation takes the forms:

$$f_s = \frac{F_s}{N} \quad \text{and} \quad f_k = \frac{F_k}{N}$$

The **angle of friction** is that at which the friction just equals the gravity component. The tangent of this angle equals the coefficient of friction.

$$f = \tan \phi \tag{44}$$

The **laws of friction** are given in Art. 99.

Values of the coefficient of friction are tabulated in reference books.

CHAPTER XIV

MACHINES

Chapter XIV treats of the application of the principles of work, power, and energy to actual machinery. Several types of machines will be analyzed, and the study of these types will make it possible for the student to apply the principles to other machines.

103. Definitions. A device that serves to modify a force or motion in amount, in direction, or in both respects, is called a machine. It is a device by which energy is received from some source outside itself and is delivered at some other point, after certain losses and transformations, there to perform some special work.

A machine has been defined as a combination of fixed and movable parts (frame, pulleys, gears, etc.) designed to transmit work to an advantage. The advantage may be in the amount of force expended, or in the distance moved, or in the direction of motion.

Consider as an illustration of a simple machine a set of pulley blocks and tackle. Let this machine be used to lift a casting onto a platform. The purpose or *advantage* of such a machine in this case is to make it possible to lift the casting with the expenditure of *less force.*

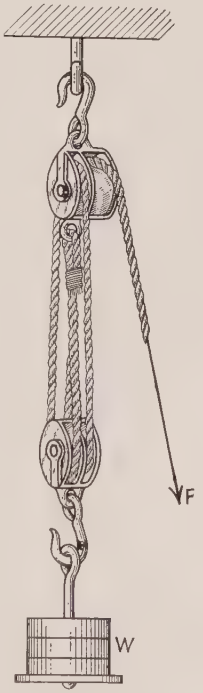


Fig. 194. Pulley block and tackle; four ropes on movable pulley.

With a block and tackle arranged so that the movable pulley holding the casting is supported by four ropes, see Fig. 194, each rope would carry one-fourth the weight of the casting, and a man would have to exert only one-fourth the force necessary to lift it directly, plus an amount to overcome the friction of the moving parts. Suppose there were no friction; this machine would have a *mechanical advantage* of 4 to 1.

Since each rope would have to be shortened a foot for each

foot the casting is lifted, the man must pull in 4 ft of rope for each foot he raises the load, giving a *displacement ratio* of 4 to 1.

Notice that he does not save himself any *work* by using this machine. He exerts one-fourth the force through four times the distance, which gives the same number of foot-pounds of work put *in* the machine as the machine puts *out* to lift the casting. In fact, more work must be put in because of the friction to be overcome. But the man does save himself in the amount of force or *effort* he puts into the machine.

It is a general law of all machines that

Input = output + work expended to overcome friction

Assuming no friction, as is sometimes convenient for purposes of computation, this equation becomes *input = output.*

The following definitions are important in connection with machines:

(*a*) The **effort** is the force applied to drive, that is, the force put into, the machine. Let it be represented by the symbol F_i.

(*b*) The **load** is the resistance overcome, or the weight lifted, that is, the force put out by the machine. Let it be represented by F_o.

In a hoisting machine, as the pulley blocks, the load equals the weight lifted and may be represented by W. The effort may then be represented by F (without a subscript).

(*c*) The **mechanical advantage** is the ratio of the load divided by the effort.

$$\text{Mechanical advantage} = \frac{F_o}{F_i} \quad \text{or} \quad = \frac{W}{F} \tag{45}$$

(*d*) The **displacement ratio** (sometimes called the velocity ratio) equals the distance that the effort moves, or acts, divided by the distance that the load moves, or acts, in the same time. Letting s_i and s_o represent the displacements of the effort and the load respectively,

$$\text{Displacement ratio} = \frac{s_i}{s_o} \tag{46}$$

(*e*) The **input** is the amount of work put into a machine. It is equal, in work units, to the effort times the displacement of the effort. Letting W_i represent the work input, the equation is

$$W_i = F_i \times s_i \tag{47}$$

(*f*) The **output** is the *useful* work delivered, or put out, by the machine. It is equal, in work units, to the load times its displacement.

$$W_o = F_o \times s_o \tag{48}$$

(*g*) The efficiency of a machine is the ratio of output to input, that is, **it equals output divided by input.** Efficiency is usually expressed in the percentage of the input that is delivered by the machine. Letting e represent efficiency,

$$e = \frac{W_o}{W_i} = \frac{F_o \times s_o}{F_i \times s_i} \tag{49a}$$

By transposition $$W_i \times e = W_o \tag{49b}$$

Thus, **input $\times$ efficiency = output.**

The above relations are expressed in mechanical units. The same relations are true in power units, or in heat units, or in electrical units.

It may be noted that a high efficiency is not necessarily an advantage. Some of the most perfected machines have a low efficiency because of their multitude of parts, or their complex arrangement. In some machines a high mechanical advantage, or an economy of operation, are more desired than efficiency.

It is often convenient to neglect friction for some parts of a computation, although it is not possible to neglect it in an actual machine. Then for a *frictionless machine*, input = output, and $F_i \times s_i = F_o \times s_o$. Transposing gives $s_i/s_o = F_o/F_i$. From the foregoing definitions (*c*) and (*d*), s_i/s_o = displacement ratio and F_o/F_i = mechanical advantage. Therefore, *when friction is neglected*, the displacement ratio = the mechanical advantage.

The displacement ratio is a matter of the design of the machine; as, for example, in the block and tackle previously discussed it depends upon the number of ropes on the movable pulley. In practice, the mechanical advantage will be somewhat less than the designed displacement ratio because friction will cause an increase in the effort (the denominator in the above fraction F_o/F_i). It is, moreover, a variable quantity depending upon the conditions of lubrication and condition of the bearings, etc.

Example 1.

If the load lifted by the machine in Fig. 195 is 64 lb, the effort 25 lb, circumference of A 48 in., and circumference of B 15 in., find (*a*) the

displacement ratio, (*b*) the mechanical advantage, (*c*) the output for one revolution of the wheel, (*d*) the input for one revolution, and (*e*) the efficiency.

Solution. (*a*) Displacement ratio $= \frac{48}{15} = 3.2$ *Ans.*

(*b*) Mechanical advantage $= \frac{64}{25} = 2.56$ *Ans.*

(*c*) Output $= 64 \times 15 = 960$ in-lb *Ans.*

(*d*) Input $= 25 \times 48 = 1200$ in-lb *Ans.*

(*e*) Efficiency $= \frac{960}{1200} = 0.80 = 80$ per cent *Ans.*

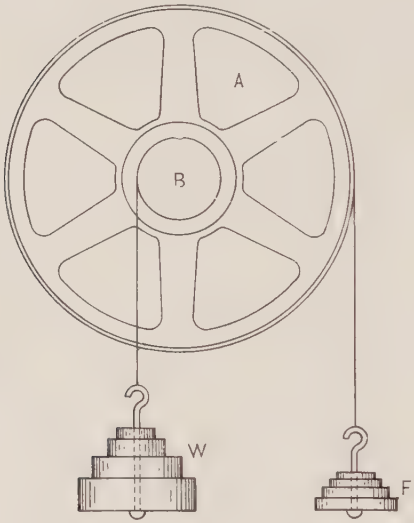


FIG. 195. A simple machine.

Example 2.

Find the output, input, and efficiency of Problem 1 if the load is lifted 1 ft.

Solution. Output $= 64 \times 1 = 64$ ft-lb *Ans.*

Displacement ratio $= \frac{48}{15} = 3.2$ to 1, i.e., the effort moves 3.2 ft for each foot moved by the load.

Input $= 25 \times 3.2 = 80$ ft-lb *Ans.*

Efficiency $= \frac{64}{80} = 0.80 = 80$ per cent *Ans.*

104. Friction of machinery. The frictional resistance of any machine and, therefore, the necessity for the expenditure of energy without the accomplishment of *useful* work may be illustrated by the study of a simple machine, as the wheel and axle of Fig. 195. To raise W with uniform velocity, it will be found that

F must be made heavy enough both to balance W and to overcome the friction of the device; or if F_t is the theoretical frictionless effort necessary to balance W, and the actual effort to *raise* W at a uniform rate is found to be F_R, then

$$F_R = F_t + F_f \tag{1}$$

When W is *lowered* with uniform velocity, it will be found that F_R may be decreased to a value F_L such that $F_t = F_L + F_f$, or

$$F_L = F_t - F_f \tag{2}$$

By adding equations (1) and (2) we have:

$$F_t = \frac{F_R + F_L}{2} \tag{50}$$

By subtracting equation (2) from equation (1), we may therefore find the value of friction at the load W, or

$$F_f = \frac{F_R - F_L}{2} \tag{51}$$

Plotting the values of the frictional resistance for different loads at W, a straight line will be obtained, as shown by Fig. 196. This line gives an intercept k on the Y axis, equal to the *friction of the unloaded machine*. Its equation is, therefore,

Friction = the friction of the unloaded machine + the increase in friction due to the load.

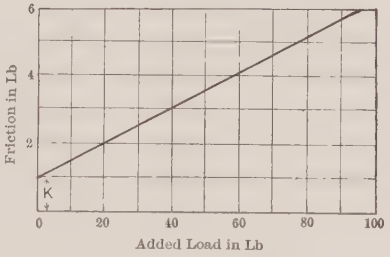


FIG. 196. Relation of friction to load on the machine.

If we designate the slope of the line by the constant c, the increase in friction with the load is cW, and

$$F_f = k + cW \tag{52}$$

Substituting values from Fig. 196, we may write the equation in the form,

$$F_f = 1 + 0.05\,W$$

Equation (52) is a perfectly general expression for friction in machines whether they are large or small, simple or complex. It should be noted, however, *that the value of friction is here measured at the circumference of the wheel.* Of course the actual resistance occurs in the bearings and is probably distributed, occurring at a number of points. It is entirely feasible to measure it in terms of the force required to turn the apparatus with a moment arm equal to the radius of the wheel. It might have been determined in terms of the force applied to the circumference of the axle, or at any other convenient point, if so desired. Thus, in machinery, a part of the driving force, wherever it is applied, is required to overcome the friction distributed through the machine, and the amount of force necessary for this purpose depends first upon the friction of the unloaded machine and second upon the increase in friction produced when load is put upon the parts. In general, therefore, friction is a disadvantage in machines, by diminishing their efficiency. In a few special machines, as for instance a screw-jack, friction is necessary for their proper operation.

105. Simple forms of machines. Levers. The simple forms of machines can be classified as modifications of the principles of the *lever* or of the *inclined plane.*

A lever is a rigid bar free to rotate about a bearing known as the fulcrum. The method of solution of most problems relating to levers will be clear from the discussion of parallel forces, Chapter V. Referring to Fig. 197, by the principle of moments,

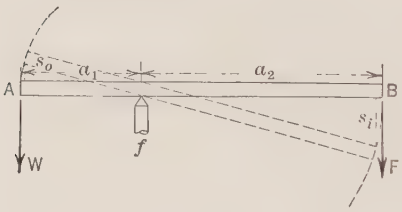


Fig. 197.

$$W \times a_1 = F \times a_2$$

Here the load, or F_o, is the weight W, and the effort, or F_i is the force F.

If, as in doing actual work, the lever rotates about the axis at f, the distances s_i and s_o through which the effort and the load move, respectively, depend upon the lengths a_1 and a_2 of the arms of the lever, and it is apparent from the similar triangles that

$$\frac{s_i}{s_o} = \frac{a_2}{a_1}$$

Therefore, neglecting friction and the weight of the lever, input = output, i.e.,

$$W \times s_o = F \times s_i$$

Velocity ratio = s_i/s_o

Mechanical advantage = W/F = same value as the velocity ratio if friction is neglected

106. The inclined plane. Consider as another class of machines an inclined plane. In a *very* simple form it may be just a smooth plank on which to slide a crate into a car. The purpose or advantage of its use is to load the crate into the car with the expenditure of less force. If a man could lift the crate directly up, that would be the simplest way; but, if the crate is too heavy for him to lift, he may resort to some form of machine that will give sufficient mechanical advantage to enable him to accomplish the result with the amount of force he can expend.

With the inclined plane the man now exerts enough effort to slide the crate up the plane and thereby puts a certain amount of work into the machine, obtaining as a result a certain amount of work from the machine, i.e., the amount of work he would have had to do if he lifted the crate vertically. The work put into the machine, measured in foot-pounds, equals the man's effort times the length of the plane. The work put out by the machine equals the weight of the crate times the *vertical* distance that it is lifted. Notice that since weight always acts vertically it is the vertical displacement and not the sloping distance that is used to get the output of the inclined plane.

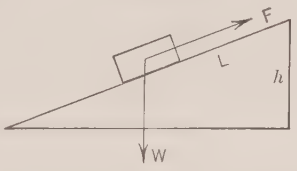


FIG. 198. The inclined plane.

If there were no friction between the sliding crate and plank, the work input would equal the work output. Referring to Fig. 198, $s_i = L$ and $s_o = h$; input = $F \times L$ and output = $W \times h$. The mechanical advantage = W F and the displacement or velocity ratio = L/h. With no friction these two terms would be equal.

107. The winch. This machine, see Fig. 199, like the wheel and axle described in a preceding article, is sometimes classed as a form of lever. The effort equals the force at right angles to the crank. The load equals the weight lifted by the rope wound around

the drum or cylinder. The velocity ratio equals the circumference of the crank circle divided by the circumference of the drum.

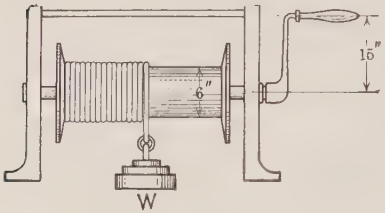


FIG. 199. Winch.

Example.

Neglecting friction, what load can be lifted by a driving force of 50 lb applied at right angles to the crank of the winch shown in Fig. 199?

Solution. In one revolution the driving force moves through a distance equal to the circumference of the circle swept out by the crank, or

$$\frac{15}{12} \times 2 \times \pi = \frac{15\,\pi}{6} \text{ ft}$$

In the same time the load is lifted a distance equal to the circumference of the axle, or $\frac{1}{2} \times \pi = \pi/2$ ft.

Therefore, $$50 \times \frac{15\,\pi}{6} = W \times \frac{\pi}{2}$$

from which $$W = 250 \text{ lb} \quad Ans.$$

108. Belt drives and gear trains. Many machines employ pulleys or gears. The computation of the velocity ratio can be accomplished by use of the equations in Chapter VIII.

Example.

Figure 200 shows a laboratory arrangement for a gear train by which the efficiency of the system may be determined.

Solution. Suppose the gear A has 80 teeth, the gear B, 60, and the gear C, 36. Suppose also that the diameter of the drum D_1 is 3 in., and of the drum D_2 is 9 in. A load is applied at W and the weight F is gradually increased until, on starting F downward, W is raised with uniform speed. Assume that when W is 200 lb, the load required at F to give uniform motion is 37.5 lb. In one revolution of C, F moves a distance equal to the circumference of D_2 or $9\,\pi$ in. When C makes one revolution, A moves forward the same number of teeth or $\frac{36}{80}$ of a revolution. The load W is therefore lifted

$$\frac{36}{80} \times 3\,\pi \quad \text{or} \quad \frac{27\,\pi}{20} \text{ in.}$$

Then the input is $37.5 \times 9\,\pi$ in-lb, and the output is

$$200 \times \frac{27\,\pi}{20} \text{ in-lb}$$

Input $\times$ efficiency = output, or

$$37.5 \times 9\,\pi \times \text{efficiency} = 200 \times \frac{27\,\pi}{20}$$

from which the efficiency is found to be 80 per cent. *Ans.*

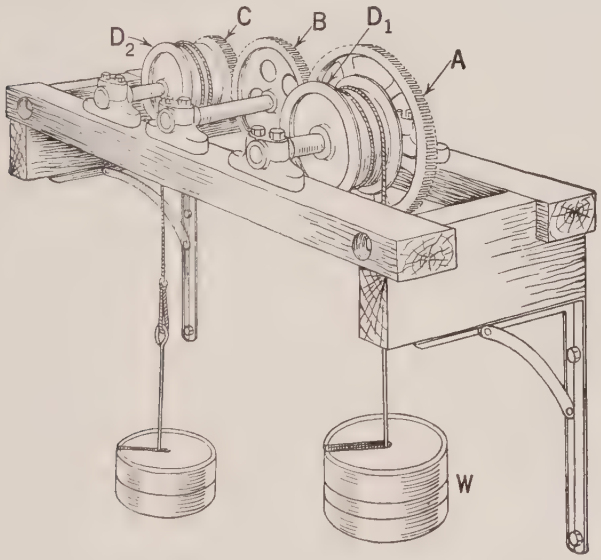


FIG. 200. Gearing arranged for laboratory test.

109. Pulley blocks and tackle. A simple pulley block or fixed pulley, arranged as in Fig. 201(*a*), is obviously of advantage only in changing the *direction* of the applied force. If we may neglect friction, $F = W$, and in raising W, the applied force F must move through the same distance that W is lifted. Arranged as in Fig. 201(*b*), F must obviously move through twice the distance that W is lifted. Hence

$$F \times 2\,s = W \times s$$

or

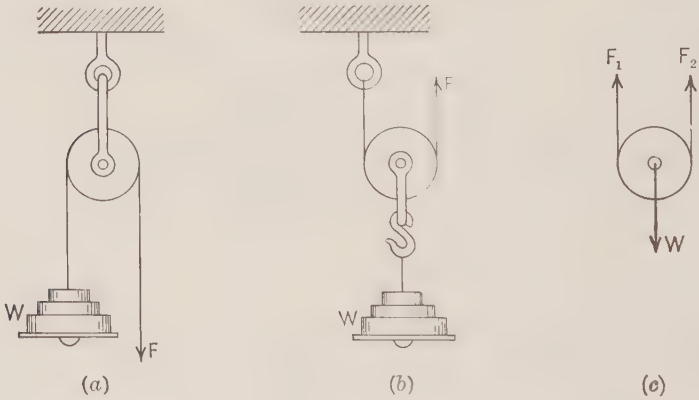
$$F = \frac{W}{2}$$

The static condition is as in Fig. 201(*c*) where neglecting friction the tension in the hoisting rope is the same each side of the block, or $F_1 + F_2 = W$, and again

$$F = \frac{W}{2}$$

In the block and tackle of Fig. 194, the displacement ratio = 4, as noted in Art. 103; and the mechanical advantage = 4, friction neglected. The input = $F \times 4$ and the output = $W \times 1$.

Different arrangements of the hoisting rope may be made according to the number of sheaves in the pulley blocks and their arrangement, but in each case where a continuous rope is used the velocity ratio may be found by the method used for Fig. 194. In any commercial block and tackle the force to lift W would be greater than this theoretical value of F because of the friction of the sheaves in the blocks and the stiffness of the hoisting rope. The actual value of F required for hoisting a given load would need to be found by experiment or from a knowledge of the efficiency of the apparatus at the given load.



(*a*) (*b*) (*c*)

Fig. 201.

110. Differential pulleys. Chain hoists. The differential **chain hoist** shown in Fig. 202, used for heavy loads, consists of two wheels or sheaves *of different diameters* rigidly fastened together, in the upper block, and one sheave in the lower block. An endless chain runs over these blocks, the rims of the sheaves being formed with "pockets" to prevent the chain from slipping upon them. The

arrangement of the sheaves and chain is shown in Fig. 205. Suppose R is the radius of the larger sheave, r the radius of the smaller. To hoist the load W, the driving force F is applied to the chain coming from the larger sheave as shown in Fig. 205. Suppose

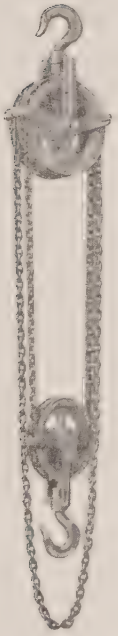


FIG. 202. Differential chain hoist.

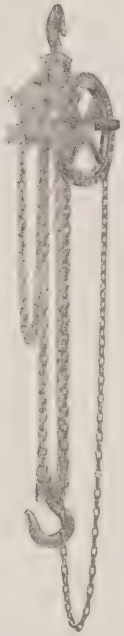


FIG. 203. Screw-geared chain hoist.

Courtesy Yale and Towne Manufacturing Company.

FIG. 204. Spur-geared chain hoist.

the upper pair of sheaves makes one revolution; then F moves a distance $2\ \pi R$, the side B of the chain is raised a distance $2\ \pi R$, but the side A is lowered at the same time a distance $2\ \pi r$, as it unwinds from the smaller sheave. The loop of chain AB is actually shortened, therefore, a distance $= 2\ \pi R - 2\ \pi r = 2\ \pi\ (R - r)$. The load W is raised one-half of this distance, or $\pi\ (R - r)$.

Therefore $$\text{Velocity ratio} = \frac{2\ \pi R}{\pi\ (R - r)} = \frac{2\ R}{R - r}$$

By making the sheaves nearly the same size (i.e., $R - r$ small), the ratio may be increased.

The energy equation for the apparatus is

$$F \times 2R \times \text{efficiency} = W \times (R - r)$$

In practice, since $2\pi R$ and $2\pi r$ equal the circumference of the large and small sheaves, respectively, it is simpler to count the pockets in each sheave. Then, letting C equal number of pockets in the large sheave and c in the small sheave, the

$$\text{Velocity ratio} = \frac{2C}{C - c}$$

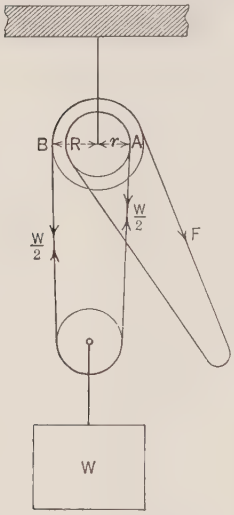


Fig. 205. Differential hoist.

One advantage of the differential hoist is that it does not run down when the force F is removed. The tendency to turn clockwise, as will be seen from the figure, is $W/2 \times r$. The counter-clockwise moment is $W/2 \times R$. The counter-clockwise moment therefore exceeds the clockwise by the difference, or $W/2\ (R - r)$. Friction of the apparatus (upper and lower blocks) must exceed this and then the load will not run down.

The chain hoist furnishes an excellent medium for a laboratory study of a "non-reversible" machine. A non-reversible machine is so called because it will not reverse itself when the effort is removed, as will the pulley blocks or wheel and axle. A *negative effort*, that is, an effort in the opposite direction, on the other side of the same pulley, is required to lower the load. As in Art. 104, half the algebraic difference of the two efforts is approximately the value of the friction for that load. Equation (51) becomes

$$F_f = \frac{F_R - (-F_L)}{2} = \frac{F_R + F_L}{2}$$

The results of a laboratory test are shown in Fig. 206, for a 1-ton spur-geared hoist, in which the efficiency is shown to be about 60 per cent at one quarter its rated full load.

Two other types of chain hoist are in common use. The *screw-geared hoist*, illustrated in Fig. 203, operates on the principle of the worm and wheel, or screw and gear, described in Art. 111. See Fig. 144. The effort or hand chain rotates the screw shaft. This rotates the gear shaft which raises the load chain. This hoist is somewhat faster and more efficient than the differential,

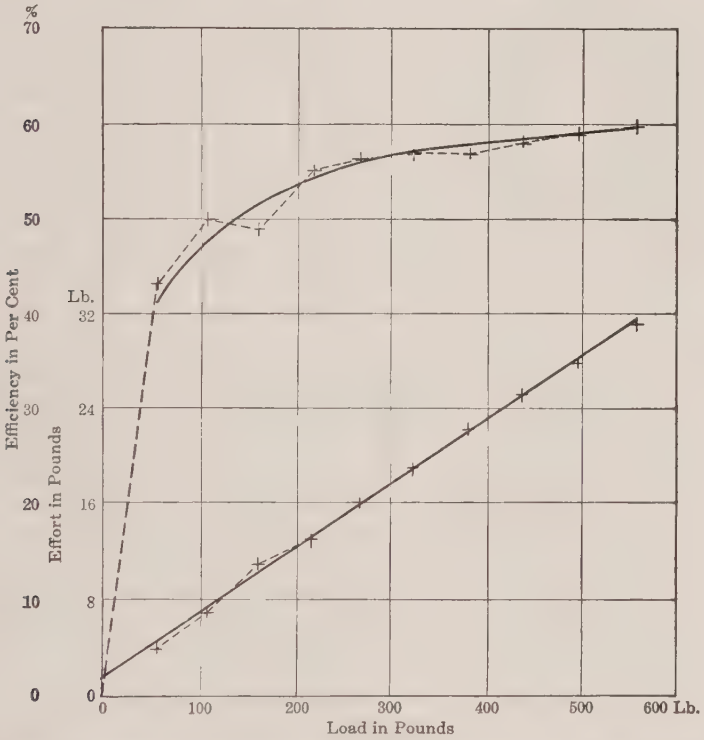


FIG. 206. Typical efficiency and effort curves from an actual laboratory test. The full lines represent an average of the actual data. The efficiency is zero, and the effort about 1.5 lb, at no load.

one manufacturer's rating being approximately 40 and 35 per cent, respectively, for full load. Like the differential hoist, it is non-reversible because of its friction, which is supplemented by an automatic brake.

The *spur-geared hoist*, Fig. 204, is the fastest and most efficient type, the manufacturer's rating being as high as 85 per cent. Friction, in this hoist, is not sufficient to prevent reversing and an

automatic ratchet brake is used to hold the load. Data on typical hoists give a mechanical advantage of about 26 to 1 for full load on a 1-ton spur-geared hoist, compared with 23 to 1 on the screw-geared and 11 to 1 on the differential hoists. The corresponding displacement ratios are approximately 31 to 1, 59 to 1, and 30 to 1, respectively.

Courtesy Yale and Towne Manufacturing Co.

FIG. 207. The spur-geared chain block with cover removed.

Figure 207 shows the spur-geared block with cover removed. The hand chain rotates the small pinion gear shown in the center. This, in turn, rotates the two diametrically opposite intermediate gears. Behind these two gears, on the same shafts, are two smaller gears (not visible in the illustration) in mesh with the large internal annular gear. Since this gear is stationary, the rotation of the pair of intermediate gears causes them to travel around, inside it, thereby rotating the yoke on which they are mounted. This yoke is on the same shaft as the load sheave which raises the load chain.

111. The screw. The screw may be regarded as a combination of the two mechanical elements, a wheel and axle, and an inclined plane. Thus, when a bolt is turned by a wrench, the circle through which the wrench swings represents the "wheel," the shank of the bolt the "axle." The thread cut on the shank adds the element of the inclined plane. In use there must be two parts: the screw and the *nut*, on the inside of which a helical thread is cut so that the thread of the screw fits in the thread of the nut. One part is rotated while the other is held so that it cannot turn, and the nut is thus made to slide axially along the screw.

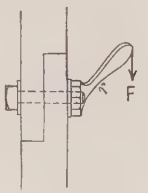


FIG. 208.

In Fig. 208 suppose the bolt A is turned by a force F applied to the wrench a distance r from the axis. In one complete turn, F therefore moves through a distance of $2\pi r$. The nut B moves in

the meantime a distance equal to the **pitch of the thread,** i.e., the distance between threads measured from corresponding places on the threads. The pitch is determined from the number of threads to the inch, and is equal to $1/N$ in., where N is the number of threads to the inch. If, for example, the bolt in Fig. 208 has 12 threads to the inch, the nut moves $\frac{1}{12}$ in. for one turn of the bolt.

The displacement ratio, then, $= \dfrac{2\pi r}{\text{pitch}}$

The input $= F \times 2\pi r$

The output $= F_o \times$ pitch (where F_o = force on the bolt)

The screw is a non-reversible machine, i.e., does not run back when the effort is removed. Its efficiency is always considerably less than 50 per cent, but, because of the large velocity ratio, a very powerful force may be exerted with a small applied effort.

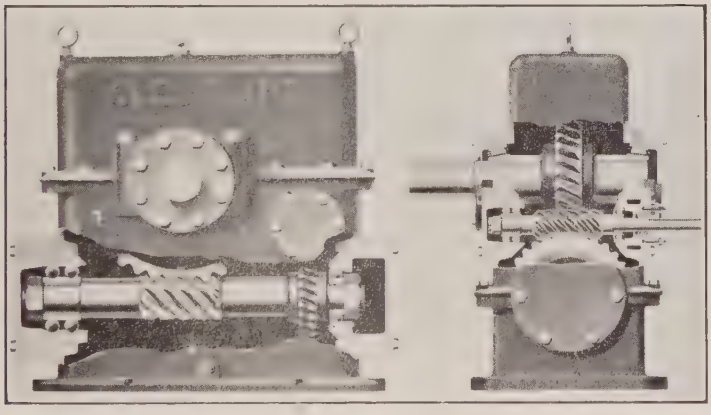


FIG. 209. Worm-reducing gear.

The **worm and worm wheel,** or endless screw, of Fig. 209, is an application of the screw employed for reducing from high to low speeds of rotation because of the large velocity ratio easily obtainable.

Another very useful application of the screw is the **screw-jack**, Fig. 210(*a*), commonly used where heavy loads are to be lifted a small distance. The velocity ratio for the screw-jack is obtained by the method explained for the screw in a preceding paragraph.

The screw-jack is another machine convenient for a laboratory study of input and efficiency. By replacing the turning bar with a grooved wheel attached to the head, the turning effort may be applied on a constant moment arm. Heavy loads on the head may be obtained by the use of a lever or any convenient device.

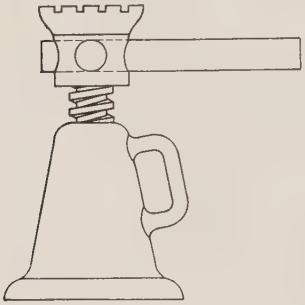


Fig. 210(*a*). Screw-jack.

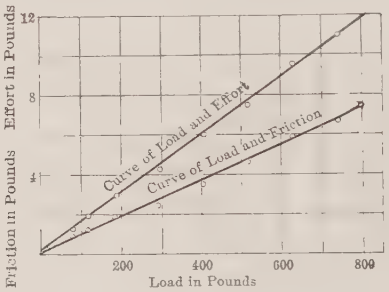


Fig. 210(*b*).

As the jack will not run back of itself, an effort must be applied to the wheel to turn it back. If F_t is the effort applied at the rim of the wheel required to lift the load under the theoretical condition of no friction, F_R the effort at the rim actually required to lift the load, and F_L the effort to turn the jack back with the same load on the head, then

$$F_R = F_t + F_f$$

$$F_L = F_f - F_t$$

From which,

$$F_t = \frac{F_R - F_L}{2}$$

and

$$F_f = \frac{F_R + F_L}{2}$$

Figure 210(*b*) shows curves plotted from experimental data with a small screw-jack. Both friction and the effort required to raise the load are shown to be proportional to the load, within the range of loads used. More then 60 per cent of the effort in raising the load was required to overcome friction, or the efficiency of the jack was less than 40 per cent.

112. Efficiency of machines in motion. Efficiency is determined by the ratio of output to input, expressing both in the same units. In an electric motor the output is usually measured in

horsepower by a Prony brake; the input is measured electrically in volts and amperes which are reduced to watts and then to equivalent horsepower. See equation (30*a*). A motor is rated by its *output* (or brake) horsepower at the load designed for maximum efficiency.

In a motor-driven machine the output may sometimes be determined conveniently by the Prony brake. If it is not convenient to equip the machine with a brake, some other means may be used to determine output. If, for example, the machine is a pump, the weight of water pumped per minute multiplied by the height lifted in feet will give the foot-pounds per minute, which

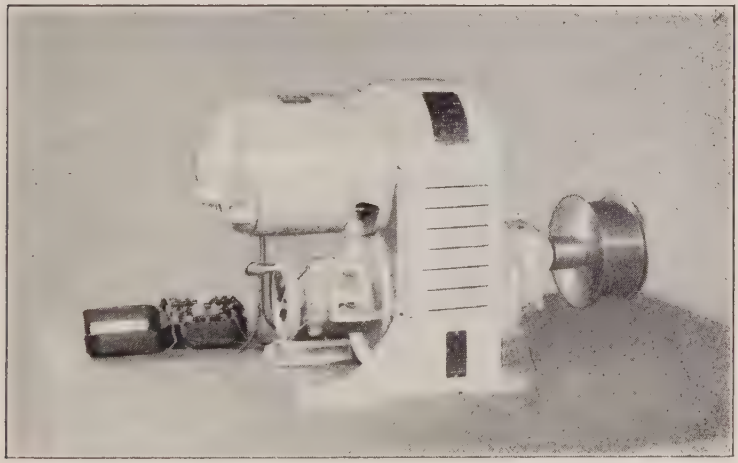


FIG. 211. Modern motor-driven gear-reduction unit, with constant-speed motor and variable-speed drive, by means of adjustable-diameter V-belt pulleys.

can be reduced to horsepower. See equation (29*a*). The *over-all* efficiency of motor and machine will be the ratio of machine output (in horsepower) to motor input (in equivalent horsepower).

To determine the efficiency of the machine alone, its input must be determined. This can sometimes be found by a transmission dynamometer. If a belt drive is used between motor and machine the loss in the belt must be allowed for. In some cases this is estimated, commonly at 10 per cent because of slip. Often a preliminary brake test is run on the motor only, from which a graph is plotted of the output for different rates of electrical input. Then the input to the machine for any given load equals

input of motor (corresponding to that load), reduced to motor output by means of the graph, and multiplied by the belt efficiency. Stated in different form, **the output of the machine equals motor input times motor efficiency times belt drive efficiency times machine efficiency.**

In a motor-driven machine in which both motor and machine are on the same shaft, or in which the shafts are in line connected by a coupling, the cradle dynamometer is most advantageous. This was described in Art. 86. The balance reading $\times$ the arm equals the output torque of the motor, which equals the input torque of the machine directly. To determine the efficiency of the machine thus requires only the computation of the power output and the ratio of this to the equivalent power input computed from the torque.

Example 1.

Given the following data from a brake test, find the efficiency of the motor. Volts = 110; amperes = 11; speed = 1800 rpm; rope tension = 12 lb on one side, 1 lb on other side; diameter of pulley = 8 in.; diameter of rope = 1 in.

Solution.

$$\text{Input} = \frac{V \times A}{746} = \frac{110 \times 11}{746} = 1.62 \text{ hp}$$

$$\text{Output} = \frac{F\pi Dn}{33{,}000} = \frac{(12 - 1) \times \pi \times (8 + 1)/12 \times 1800}{33{,}000} = \frac{11 \times \pi \times 9 \times 1800}{33{,}000 \times 12} = 1.41 \text{ hp}$$

$$\text{Efficiency} = \frac{1.41}{1.62} = 0.87 = 87 \text{ per cent} \quad \textit{Ans.}$$

Example 2.

Find the output of a machine, belt driven from a motor, and the over-all efficiency, given the following data. Motor input = 1000 watts; motor efficiency = 85 per cent, belt slip 10 per cent, machine efficiency = 65 per cent.

Solution.

$$\text{Input of motor} = \frac{1000}{746} = 1.33 \text{ hp}$$

$$\text{Output of machine} = 1.33 \times 0.85 \times 0.90 \times 0.65 = 0.66 \text{ hp.} \quad \textit{Ans}$$

$$\text{Over-all efficiency} = \frac{0.66}{1.33} = 0.50 = 50 \text{ per cent} \quad \textit{Ans.}$$

PROBLEMS

14–1. A weight of 500 lb is lifted 9 ft by means of a set of triple pulley blocks and tackle, with which there are six ropes supporting the pulley that holds the load. Neglecting friction find (*a*) the displacement ratio, (*b*) the effort, (*c*) the mechanical advantage, (*d*) the output, (*e*) the input.

14–2. Referring to Fig. 194, what is the displacement ratio of the blocks and tackle as arranged? If the tackle were laid out horizontally in order to haul a truck out of the mud and the upper block attached to the truck, what would be the ratio? Assuming the resistance offered by the truck to be 1000 lb, how much difference in pounds effort (friction neglected) would be made by attaching the upper instead of the lower pulley block to the truck?

14–3. If a man pulls 75 lb on the block and tackle of Fig. 194 to lift a weight of 200 lb, what is the efficiency?

14–4. A man lifts a 200-lb load 5 ft by pulling out 25 ft of rope. Assuming the block and tackle to be 80 per cent efficient, how hard does he pull?

14–5. A weight of 300 lb is lifted by a rope wound around a shaft of 2-in. diameter, by a pull of 50 lb on the rim of a 15-in. diameter wheel on the shaft. What is the efficiency?

14–6. A safe weighing 5 tons is to be loaded on a truck 4 ft high by means of planks 18 ft long. If it requires 200 lb to overcome the friction of the safe on the planks, find the least force that will be necessary to move the safe.

14–7. If with the winch of Fig. 199 a force on the crank handle of 80 lb is required to lift a load of 320 lb, what is the efficiency of the machine and what effort is used to overcome friction alone?

14–8. The pilot wheel of a boat is 3 ft in diameter; the axle is 6 in. The resistance of the rudder is 180 lb at the axle. What effort applied to the wheel will move the rudder? (Neglect friction.)

14–9. Four men are hoisting an anchor weighing one ton. The barrel of the capstan is 8 in. diameter, the length of the hand spikes 3 ft 4 in. (from center of barrel). How much force must each man exert? What is the velocity ratio?

14–10. In an arrangement of gears, as in Fig. 200, A has 72 teeth, B has 60 teeth, C has 32 teeth. The drum for F is 6 in. in diameter, that for W, 4 in. in diameter. Find the efficiency of the apparatus and the force required at F to overcome friction when a force of 30 lb at F will raise a load of 85 lb.

14–11. In the worm and worm wheel of Fig. 144, the wheel has 98 teeth. If the worm were on a shaft turning 2400 times per minute, how many turns per minute would the wheel make? If used as a hoisting mechanism, the drum B being 8 in. in diameter and the worm being turned by a handle 18 in. long, what is the velocity ratio? If 50 per cent efficient what load can be lifted by an effort of 100 lb?

14–12. In a letter press, the diameter of the wheel is 14 in.; the pitch of the thread is $\frac{3}{16}$ in. Find the pressure applied by a pull of 50 lb tangent to the wheel, if the efficiency of the press is 30 per cent.

14–13. The diameter of the pulley A in Fig. 145 is 16 in., of the pulley B, 10 in. Drum C is 12 in. in diameter and drum D, 3 in. If 5 lb are required at F to overcome friction when W is 90 lb, what value of F will raise the 90 lb with uniform speed? What is the efficiency of the apparatus at this load?

14–14. In moving a building, the horse is attached to a lever 7 ft long acting on a capstan barrel 11 in. diameter; on the barrel winds a rope of a system of 2 fixed (near capstan) and 3 movable pulleys. What force will be exerted when the horse pulls 500 lb, allowing 50 per cent loss from friction?

14–15. An inclined plane 9 ft long rises 2.5 ft. A box weighing 75 lb is pulled up by a force of 32 lb. What is the efficiency?

14–16. If a load of 1 ton can be lifted $\frac{1}{4}$ in. on a screw-jack by pulling 28 lb on a bar 16 in. long (from center of screw) through one complete circle, what is the efficiency?

14–17. What load can be raised by a 50-lb pull on a 28-in. bar (measured from center of screw) on a screw-jack with a $\frac{3}{8}$-in. pitch, allowing an efficiency of 20 per cent?

14–18. In an ordinary screw-jack, the screw is turned by a lever 2 ft long (from center of screw). If there are 4 threads to the inch, find the force required on the end of the lever to lift 10,000 lb, when 65 per cent of the total effort is required to overcome friction.

14–19. By experiment with a chain hoist it was found that a pull of 15 lb on the leading side was required to lift 60 lb, including the lower pulley. The larger pulley in the upper block was 4 in. diameter, the smaller, $3\frac{1}{2}$ in. Find the efficiency at this load. (See Fig. 205.)

14–20. Find the efficiency of a differential chain hoist having 26 pockets on the large and 25 on the small wheel, if a pull of 50 lb will lift $\frac{1}{2}$ ton.

14–21. A differential hoist has 30 pockets on the large and 29 on the small wheel and is rated at 35 per cent efficient. What effort will lift a load of 2000 lb?

14–22. If the circumference of the hand-chain sheave of a 1-ton screw-geared hoist is 22 in., the circumference of the gear sheave which raises the load 10 in., and the number of teeth in the gear 25, find (*a*) the displacement ratio, (*b*) the mechanical advantage at full load if the efficiency is 40 per cent.

14–23. The hand chain moves 30 in. for each revolution of the central drive pinion of a spur-geared chain hoist, Fig. 207, and the load chain 10 in. for each revolution of the load sheave. The load sheave rotates once for each 10 revolutions of the small central drive pinion. Find (*a*) the displacement ratio, (*b*) the load that can be lifted by an effort of 80 lb, assuming an efficiency of 85 per cent, (*c*) the mechanical advantage.

14–24. A railway engine hauls a train on a level track at a speed of 30 mph against an average resistance of 4500 lb. It burns 960 lb of coal per hour. If the combustion of each pound of coal supplies 10,000,000 ft-lb of energy, compute input, output, and efficiency of the engine.

SUMMARY

A machine is defined as a device that serves to modify a force or motion in amount, in direction, or in both respects. It has been defined also as a combination of fixed and movable parts designed to transmit work to an advantage.

The general law of machines is that Input = Output + work expended to overcome friction.

The effort is the force put into the machine to drive it = F_i or F.

The load is the force put out by the machine = F_o or W.

The mechanical advantage = load ÷ effort = F_o/F_i or W/F. (45)

The displacement or velocity ratio = distance that effort moves ÷ distance that load moves = s_i/s_o. (46)

The input is the **work** put into the machine.

$$W_i = F_i \times s_i \tag{47}$$

The output is the **work** put out by the machine.

$$W_o = F_o \times s_o \tag{48}$$

Efficiency = Output ÷ Input, expressed in per cent.

$$e = \frac{W_o}{W_i} \tag{49a}$$

Input × efficiency = output. $W_i \times e = W_o$. (49b)

The theoretical frictionless effort = ½ the sum of the effort to lift + effort to lower (uniformly).

$$F_t = \frac{F_R + F_L}{2} \tag{50}$$

The friction = ½ the difference of the effort to lift − the effort to lower (uniformly).

$$F_f = \frac{F_R - F_L}{2} \tag{51}$$

If the machine is non-reversible, the effort to lower is negative.

Friction = the friction of the unloaded machine + the increase in friction due to the load. $F_f = k + cW$ (52)

Efficiency may be found from the ratio of output to input in any units, provided both are in the same units. Common units are foot-pounds, horsepower, British thermal units, watts.

The relative values of a series of **progressive steps in machinery** can be found from the intermediate efficiencies, thus: Output of a motor-driven machine = motor input × motor efficiency × belt-drive efficiency × machine efficiency.

CHAPTER XV

ELASTICITY AND STRENGTH OF MATERIALS

This chapter can be studied immediately following Chapter VII if so desired. It treats mostly of the application of the principles of statics, that is, of forces in equilibrium at rest, to the mechanics of materials. Two important properties of materials, in respect to mechanics, are elasticity and strength. In practice, the elasticity and strength of materials must be considered for bodies both at rest and in motion, as in machinery.

After the external forces acting on a body have been determined, the stresses within the fibres are considered. If the body is at rest, the external forces are determined by the methods of Chapters I to VII. If the body is in motion, or is acted on by bodies in motion, the relations of force, mass, and acceleration, in Chapters VIII to X, are used together with the friction and energy forces treated in Chapters XI to XIV.

After the internal stresses are computed, then the selection of the most suitable material is made, and the size and shape of members designed to give sufficient strength and elasticity to be safe and satisfactory. If the machine or structure is already constructed, it can be measured, and an estimate made of the loads it can safely carry without failure or undue deflection of form.

113. Elasticity. In our study of bodies in equilibrium and in motion we have assumed that the bodies were rigid. As a matter of fact, the application of external forces always produces a change in the form of a body, the amount and character of such change being dependent upon the amount and arrangement of the applied *forces*, and upon the *material* of which the body is composed.

Elasticity may be defined as the property of bodies by virtue of which they resume their original form when the external forces acting on them are removed.

As a simple illustration, consider a wire, as AB, Fig. 212, attached to a rigid support at A and carrying a scale-pan at B. By adding suitable weights to the scale-pan, we may subject the wire to a pull of any desired amount. To determine the effect,

Suppose we place the wire in slots in the guides GG to prevent swinging, and then fasten a thread attached to the short arm of a light pointer P, moving freely about an axis at D, to a point on the wire by means of a small spring clip C.* If the wire stretches, the clip C will be carried downward, the pointer will be rotated about the axis, and the long arm will be moved along the steel scale S by an amount equal to $\frac{\text{length of long arm}}{\text{length of short arm}} \times$ the distance C moves.

To correct for errors in our experiment arising from the sagging of the support at A, etc., a second pointer is attached in the same manner to a point on the wire near A. The length of wire under investigation is thus limited to the portion between the clips C and C', and the actual stretch produced in this length by any given load at L is measured by the amount that the *lower clip moves more than the upper*. By this arrangement, a wire 8 ft or more in length may be used conveniently in an ordinary room. With pointers 10 in. from the scale to the axis D and 1 in. from D to the center of the hole where the thread from the clip is attached, and scales on which the positions of the pointers may be read to $\frac{1}{100}$ in., the stretch in 8 ft of wire can be measured to about $\frac{1}{1000}$ in., or about $\frac{1}{8000}$ in. in each inch of length. Closer readings should not be attempted.

FIG. 212. Apparatus for elasticity test.

In performing our experiment, suppose we first apply a load of 10 or 20 lb to remove any kinks from the wire, and then, having first set each pointer near the bottom of its scale and noted its position, suppose we add weights in 5-lb steps to the scale-pan, and read the positions of the pointers after each increase in load. Since we are concerned for the moment only with the effect of a given increase in load, we may disregard the effect of the straightening load and assume that we start from zero load.

Readings for a piano steel wire 0.0348 in. in diameter and 90 in. long between clips are shown in the table on page 236.

* The ordinary triangular spring wire clips used for fastening together loose sheets of paper have proved perfectly satisfactory for this purpose.

Added Load, Lb	Position of Pointers		Total Apparent Stretch	Total Actual Stretch	Stretch per Inch. (Unit deformation)	Load in Lb per Sq In. (Stress)
	Upper	Lower				
0	1.14	0.16				
5	1.19	0.37	0.16	0.016	0.00018	5,300
10	1.21	0.56	0.33	0.033	0.00037	10,500
15	1.23	0.73	0.48	0.048	0.00053	15,900
20	1.24	0.90	0.64	0.064	0.00071	21,200
25	1.25	1.06	0.79	0.079	0.00088	26,500
0	1.23	0.25	0.00	0.000	0.00000	00,000

From these figures, it is apparent that the wire when supporting a scale-pan and weights was longer than the same wire with no load; in other words, a pull caused the wire to stretch. Also, that the wire continued to stretch more and more as the load increased but immediately returned to its original length when the stretching load was removed. This is the characteristic property of **elastic material**; therefore we may say that the steel wire was elastic under the conditions present in the experiment. (The permanent change in the positions of the pointers was due to sagging of the support, stretch in the wire above the upper pointer, etc., as was evidenced by the fact that both pointers remained displaced by the same amount.)

114. Stress. When a load is placed in the scale-pan of the apparatus, Fig. 212, a pull equal to the total weight of the scale-pan and its contents is applied to every section of the wire, tending to separate the particles of steel along this section from those next above them. This result is opposed by the cohesion between the pairs of particles, and thus the pull is transmitted through the wire to the support. **The internal resistance which balances an external force is called a stress.** See Art. 8.

The total pull at a section is distributed over all the pairs of particles along that section. The total stress is equal to the total externally applied force and is expressed in the same units. If we divide the total stress by the area of section in square inches, we shall have the stress per unit of area to which the material is subjected. This is called the **unit-stress.**

$$\text{Unit-stress} = \text{stress per square inch} = \frac{\text{total stress}}{\text{area of section in sq in.}}$$

Letting s = unit-stress, F = total load = total stress, and A = area in sq in., then:

$$s = \frac{F}{A} \tag{53}$$

Thus when the total exterior pull is 10 lb, the *tensile stress* in the wire is 10 lb and the *unit-stress* is $\frac{10}{(0.0348)^2 \times 0.7854}$, or 10,500 lb per sq in. If the cross section of the wire is not uniform, the greatest unit stress occurs where the diameter is least.

A direct stress may be of three kinds: **Tensile,** as in the preceding illustration; **compressive,** as in columns and struts supporting a load, in which case the particles are pushed closer together, and the tendency is to crush the material; or **shearing** when one portion of the material tends to slide past another, as in the relative motion of the blades of a pair of shears. A shearing stress is set up in punching holes in a plate, in bending beams, etc. See Art. 119.

115. Deformation. Hooke's Law. When a body is put under stress by the application of exterior forces, a deformation, or change in the shape, of the body is produced. This deformation has a definite relation to the stress.

Hooke's Law states that a body acted upon by an exterior force has a deformation proportional to the stress, as long as the elastic limit is not exceeded.

The amount of deformation is expressed by determining **the unit-deformation, i.e., the stretch or shortening per unit of original length.** Thus in the experiment with a piano steel wire, Art. 113, a load of 20 lb produced a stretch of 0.064 in. in a wire 90 in. long; the deformation produced was therefore 0.064 90 or 0.00071. Similarly, if a column 10 in. long shortens under compression until only 9.98 in. long, the total shortening is 0.02 in., and the unit shortening or unit-deformation is 0.02/10.00 or 0.002. As the unit-deformation for the wire under tension, or the column under compression, is simply the ratio of two lengths, the change in length and the original length, it is an abstract number.

116. The elastic limit. This term signifies the maximum unit-stress to which a material may be subjected and still return to

its original shape when the stress is removed. Hooke's Law is true practically up to this point, although careful data indicate that the *limit of the proportion* of deformation to stress occurs at a unit-stress *just slightly below the elastic limit*. At a unit-stress *slightly higher* than the elastic limit occurs the **yield point, at which the deformation first starts to increase more rapidly without an appreciable increase in the load.** This point is indicated by a positive drop in the weighing beam of the testing machine. In general, the term elastic limit is used to refer to any and all of these values as they are so nearly the same for most materials, particularly the metals.

If we plot the curve from the data of the last two columns of the table in Art. 113, we obtain the curve of Fig. 213. This is a straight line passing through the origin, thus showing that, for the range of stresses produced by the loads applied, the *stretch was directly proportional to the stress*. We note, also, that the wire

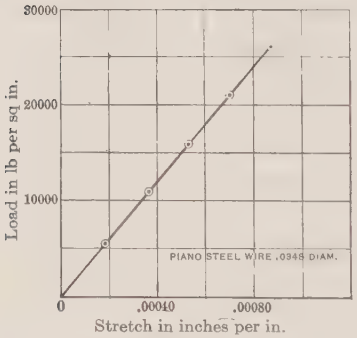


Fig. 213. Curve showing the relation of stretch to load for a piano steel wire.

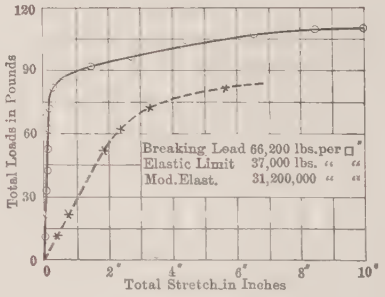


Fig. 214. Deformation diagram for annealed steel wire.

returned immediately to its original length when the largest load used was removed. Suppose, however, to avoid the necessity for inconveniently large loads, we repeat the experiment using instead of the piano steel wire one of annealed steel 0.048 in. in diameter and 86 in. long. The stretching loads were applied in the same manner as before, but were increased, step by step, until the wire broke at 109 lb actual pull, after a total elongation of about 10 in. The full line of Fig. 214 shows the complete history of the test; the dotted curve shows the results for loads below 85 lb with, for

clearness, the stretch plotted to twenty times the scale used for the full line.

This dotted curve is a straight line up to a pull of about 45 lb or 27,500 lb per sq in. Tests made within this range also showed that the annealed steel wire returned to its original length when the stretching load was removed. As the load was increased beyond 45 lb, however, the curve began to *deviate slightly* from a straight line, and after about 60 lb actual pull or 37,500 lb per sq in., the elongations increased in a perceptibly *faster ratio than the loads*. Moreover, when the load was removed, the wire failed to spring back to its original length, but remained permanently stretched, as it had taken a *permanent set*.

117. Modulus of elasticity. The ratio $\dfrac{\text{unit-stress}}{\text{unit-deformation}}$ is known as the **modulus of elasticity** for a material (sometimes called the *coefficient of elasticity*).

For the piano steel wire of our experiment, the modulus of elasticity in tension is $\dfrac{\text{unit-stress}}{\text{unit-deformation}} = \dfrac{21{,}000}{0.00071} = 30{,}000{,}000$ nearly, which is the accepted value for most grades of steel.

The modulus of elasticity in compression may be computed in a similar way for short pieces or for longer ones which are prevented from bending. The piece is subjected to a known load tending to shorten it, and the amount of decrease in length is measured. From the load and area of cross section, the unit-stress is computed, and from the total shortening and the original length, the unit-deformation is found. The modulus of elasticity in compression has practically the same value as the modulus of elasticity in tension, or 30,000,000. Since the unit-deformation is an abstract number, the modulus is expressed in the same units as the unit-stress, that is, in pounds per square inch. Within the elastic limit, the modulus of elasticity is a constant.

The modulus of elasticity is expressed, mathematically, by the equation

$$E = \frac{\text{unit-stress}}{\text{unit-deformation}} = \frac{\dfrac{F}{A}}{\dfrac{e}{L}} = \frac{FL}{Ae} \quad \text{and} \quad e = \frac{FL}{AE} \qquad (54)$$

in which E is the modulus, F the load in pounds, A the area of

cross section in square inches, e the total elongation or shortening, and L the original length of the piece.

Example.

How much will a wrought-iron bar 1 in. by $1\frac{1}{2}$ in. in section and 5 in. long shorten under a load of 30,000 lb?

Solution. From the table, Art. 118, E for wrought iron = 25,000,000.

Therefore, $e = \dfrac{30{,}000 \times 5}{1.5 \times 25{,}000{,}000} = 0.004$ in. *Ans.*

118. Ultimate strength. When any member of a structure is under stress exceeding the elastic limit of the material of which it is composed, the member is usually unsafe. If the stress is increased, the deformation rapidly increases, and if the stress exceeds a certain limit, the rupture or failure of the member will occur.

The **ultimate strength** of any material in tension, compression, or shear is the unit tensile, compressive, or shearing stress required for rupture. Thus, if a wrought-iron bar $\frac{3}{4}$ in. diameter breaks at an actual pull of 28,700 lb, its ultimate strength for tension is

$$s = \frac{F}{A} = \frac{28{,}700}{(\frac{3}{4})^2 \times 0.7854} = 65{,}000 \text{ lb per sq in.}$$

The ultimate strengths of the ordinary materials of construction for simple tensile or compressive stress have been pretty definitely determined. Ultimate strengths for shear and for cases in which the material is subjected simultaneously or in alternation to more than one type of stress are not so definitely known. Safe loads for such conditions are fixed largely by practical experience.

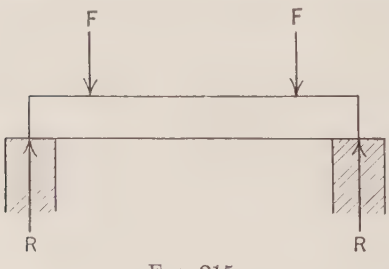


FIG. 215.

119. Shear. Shearing stresses are set up when two forces act like a pair of shears tending to cut a body between them. Thus in the beam of Fig. 215, there are near each support two parallel forces which tend to cut the beam across vertically. As illustrated, $R = F$, and the shearing stress in a section of the beam near a support is equal

to F. The deformation is a vertical sliding between the forces upward and downward. When two bars are riveted together and then subjected to tension, the section of the rivet between the bars is under shear. If a bolt is in tension, there is a tendency to shear the head of the bolt from the shank.

AVERAGE VALUES FOR THE ELASTIC LIMITS, ULTIMATE STRENGTHS, AND MODULUSES OF ELASTICITY FOR COMMON STRUCTURAL MATERIALS

Material	Elastic Limit (Lb per Sq In.)		Ultimate Strength (Lb per Sq In.)			Modulus of Elasticity for Tension and Compression (Lb per Sq In.)
	Tension	Compression	Tension	Compression	Shear	
Timber	3,000	3,000	10,000	8,000	3,000 (Across grain)	1,500,000
Cast iron	6,000	20,000	20,000	90,000	20,000	15,000,000
Wrought iron	25,000	25,000	50,000	50,000	40,000	25,000,000
Structural steel	35,000	35,000	65,000	65,000	50,000	30,000,000

Example.

An iron bolt $1\frac{1}{4}$ in. in diameter and having a head 1 in. long is subjected to a pull of 10,000 lb. Find the unit tensile stress; also the unit shearing stress.

Solution. The section of the bolt is $.7854 \times (1\frac{1}{4})^2$ or 1.23 sq in., and the unit tensile stress on the shank is 10,000/1.23 or 8130 lb per sq in. *Ans.*

The head tends to shear off along the surface of a cylinder $1\frac{1}{4}$ in. in diameter and 1 in. long. The shearing area is therefore $3.1416 \times 1\frac{1}{4} \times 1$, or 3.93 sq in. and the unit shearing stress on the head is 10,000/3.93 or 2520 lb per sq in. *Ans.*

PROBLEMS

15–1. Find the stress in a $\frac{7}{8}$-in. diameter rod held vertically, from which is suspended a load of 7000 lb.

15–2. Find the pull that can be carried by a bar $\frac{3}{4}$ in. $\times$ 3 in. to give a stress of 15,000 lb per sq in.

15–3. If a $\frac{5}{8} \times 1\frac{3}{4}$ in. bar of alloy metal breaks under a pull of 34,000 lb, what load would probably break a bar $\frac{3}{4} \times 2\frac{1}{2}$ in.?

15–4. A bolt is $1\frac{1}{2}$ in. in diameter. If subjected to a pull of 12,500 lb, what is the unit-stress?

15–5. A wrought-iron tension member of a truss is to be designed for a unit-stress not to exceed 7500 lb per sq in. when the member is subjected to a pull of 60,000 lb. What must be the area of cross section?

15–6. A rod 28 ft long and $1\frac{3}{4}$ in. in diameter is stretched $\frac{3}{16}$ in. by a load of 40,000 lb. Compute the unit-stress, unit-deformation, and modulus of elasticity.

15–7. A steel eye-bar is 30 ft long and $1\frac{1}{2} \times 6$ in. in section. How much does it elongate under a pull of 90,000 lb?

15–8. Find the modulus of elasticity of steel from the following data: length of wire, 60 ft; diameter, .04 in.; load, 12 lb; elongation, 0.24 in.

15–9. What is the modulus of elasticity of the material when a bar of 1 sq in. cross section and 20 in. long stretches 0.008 in., due to a pull of 10,000 lb?

15–10. What will be the elongation of a steel bar 24 ft long under a load of 24,000 lb., if it measures 1×3 in. in cross section?

15–11. What tensile load on a $\frac{3}{8}$-in. diameter bar of steel will cause a stress equal to the elastic limit?

15–12. What load on a bar $\frac{3}{4}$ by $1\frac{1}{2}$ in. will just reach the elastic limit of steel?

15–13. What force will be necessary to shear off a bar of steel $1\frac{1}{8} \times \frac{5}{8}$ in.?

15–14. Find the compressive stress in a concrete footing 14 in. square under a load of 100 tons.

15–15. Find the dimensions of a square block of yellow pine to support a load of 36,000 lb, if the allowable stress must not exceed 1000 lb.

15–16. A round wrought-iron rod $2\frac{1}{2}$ in. in diameter broke under a pull of 271,000 lb. Compute its ultimate tensile strength.

15–17. A steel column 6 sq in. in section and 8 in. long bears a load of 75,000 lb. How much will it be compressed?

15–18. What pull will be required to break a rod of structural steel 2 in. in diameter?

15–19. A plate $\frac{9}{16}$ in. thick has a shearing strength of 60,000 lb per sq in. What force will be required to drive a punch $\frac{3}{4}$ in. in diameter through the plate?

15–20. A steel tie rod is $1\frac{1}{4}$ in. in diameter. What load can be applied to the rod if the unit-stress is not to exceed one-half the elastic limit?

15–21. A wrought-iron bolt $1\frac{1}{2}$ in. diameter has a head 1 in. long. When a tension of 16,500 lb is applied to the bolt, what is the unit tensile stress on the shank? The unit shearing stress on the head?

15–22. If the tension on the bolt of Problem 15–21 is increased until the bolt fails, will it fail through shearing off the head or through rupture of the shank?

120. Repeated stresses. The law, if any, governing the change in the elastic properties and strength of a material brought about by stressing it beyond its elastic limit, or by stressing it repeatedly below its elastic limit, has long been an interesting and important question in engineering practice. The ordinary formulas of the mechanics of materials have been based on an assumption that the

material is homogenous and that the stress is regularly distributed throughout the material. Moreover, experimental determinations of the average values of the elastic limits and ultimate strengths of the common materials of construction have been based on tests in which the load has been increased continuously in one application, until failure occurred. The increasing use of materials under conditions in which they are subjected to repeated stresses of the same kind, often enormous in number, and to repeated stresses of different kinds, as in automobiles, airplanes, machine frames, etc., makes a knowledge of the effects of such stresses of great importance.

It has long been known that repeated hardening and annealing seems to strengthen steel, and that cold-drawn pieces which are reduced in size by repeated rolling when cold between polished rollers show, on test, greater ultimate strength and a higher elastic limit than the original material. In the process there seems to be some rearrangement of particles initially disadvantageously placed so that these particles are in a position to exert an increased resistance to breaking. Specimens taken from the interior of large, solid shafts give usually much lower values for strength and elastic limit than those obtained with specimens taken nearer the circumference, thus showing the effect of difference in working and tempering. Metal which has been loaded beyond its elastic limit and then allowed to rest commonly shows on retest a higher elastic limit and greater ultimate strength, but less ductility. The curves of Fig. 216 were obtained in such a test upon a Bessemer steel bar 18 in. long and $\frac{3}{4}$ in. diameter. The loads were applied by an Olsen testing machine, and elongations taken with an extensometer. After each test in which the stress was increased beyond the elastic limit, the specimen was allowed to rest for two days before retesting, except for the fifth and last test, which was continued until the specimen broke. The repeated stresses beyond the elastic limit had increased the ultimate strength of the material about 15 per cent and had raised the elastic limit, but the ductility of the ma-

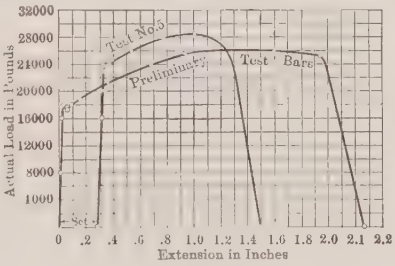


Fig. 216.

terial was decreased until the elongation was nearly 34 per cent less.

On the other hand, both experience and experiment have demonstrated that rupture may be caused by a unit-stress less than the ultimate strength when the stress is applied a great number of times in succession. Apparently, repeated stress may produce two effects upon the strength of a metal. The first of these is a strengthening effect such as noted in Fig. 216. This has been explained as due to a better adjustment of bearing between the particles of the material which may be likened to the packing down of a loose material, as sand, when subjected to jarring, thereby rendering the mass more stable.

In contrast to this strengthening effect of repeated stress, there is a tendency for a second effect which is destructive. As a result, rupture may finally occur at a unit-stress much less than that necessary before the repeated stresses had been applied. It has been suggested that this destructive effect of repeated stresses may be due to the progressive spread of minute fractures, which act something like minute saw cuts, gradually spreading because of the intensified stress at their edges, until the cross section of the metal piece is so reduced that it cannot withstand the applied working load.*

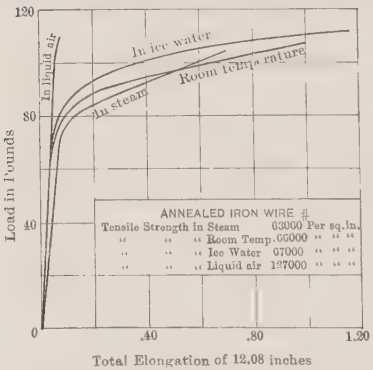


FIG. 217. The effect of temperature on the strength of annealed iron wire.

121. Effect of temperature on the strength of materials. The results of tests, conducted in the Physics Laboratories of Pratt

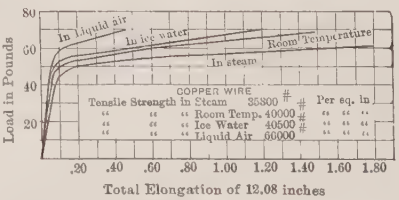


FIG. 218. The effect of temperature upon the strength of copper wire.

Institute, of the relative ultimate strength of iron and copper at room temperature (70 F), at steam temperature (212 F), in ice

* University of Ill. Eng. Exp. Sta. Bul. 142.

water (about 40 F), and in liquid air (−312 F) are shown in Figs. 217 and 218. These curves show that both the ultimate strength and the elastic limit are greatly affected by temperature. The tensile strength of iron was found to be about 95 per cent greater at −312 F than at 70 F, and that of copper about 50 per cent greater.

These results are shown even more strikingly in the curves of Fig. 219 in which tensile strength is plotted against *absolute temperatures* fahrenheit. From these curves it appears that the tensile strength of both iron and copper tends to approach a minimum (which for iron is at about the temperature of boiling water), after which it again increases. These results are borne out by tests made at the Watertown Arsenal, in which it was found that for wrought iron a minimum tensile strength is reached at a temperature between 200 F and 300 F (660 and 760 absolute), after which the strength again increases as the temperature rises until about 600 F when the strength is 20 per cent to 25 per cent above normal. From this temperature the tensile strength decreases rapidly with rise in temperature.

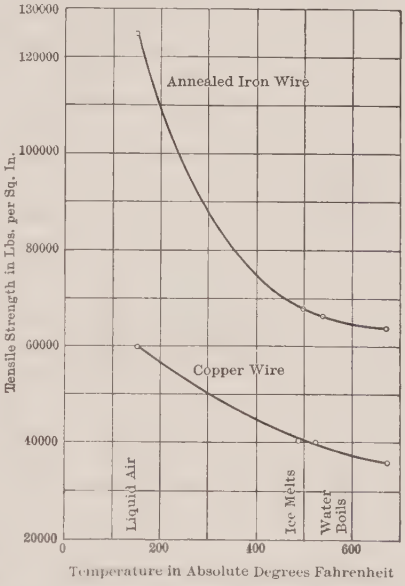


FIG. 219.

122. Stresses due to temperature. A rod free to move elongates when its temperature is raised and shortens when its temperature is lowered. If the ends of the rod are fixed, the change of temperature produces a certain unit-stress which is the same as that which would cause a change in length equal to that produced by the change in temperature in the rod free to move.

The coefficient of linear expansion, C, is defined as the change in length of each unit of length for a rise of temperature of one degree. The elongation e of a rod of unit length for a temperature

change of t degrees is, therefore, expressed by the equation $e = Ct$. Since the modulus of elasticity in tension $E = \dfrac{\text{unit-stress}}{\text{unit-elongation}}$,

$$s = eE = CtE \tag{55}$$

If the temperature of a rod fixed at both ends rises, the stress s is compression, if it falls, the stress s is tension.

Example.

To pull together two walls of a building they are connected by a wrought-iron tie rod $1\frac{1}{2}$ in. in diameter. The rod is then heated and the nuts at each wall turned up until the rod is tight. If the temperature of the rod falls 100°, what pull due to the temperature change is applied to each wall?

Solution. The coefficient of linear expansion for wrought iron is 0.0000067 and the modulus 25,000,000. Therefore, unit-stress due to temperature change = $0.0000067 \times 100 \times 25{,}000{,}000 = 16{,}800$ lb per sq in. The section of the rod is 1.77 sq in.; therefore the total pull is $1.77 \times 16{,}800 = 29{,}700$ lb. *Ans.*

123. Factor of safety. In the design of any machine or member of a structure, the maximum unit-stress to which a part may be subjected in use must always be less than the ultimate strength of the material of which the part is composed, otherwise rupture or failure of the part will occur. It must also be less than the elastic limit to avoid permanent deformation. For safety, good design provides that the working unit-stress, or unit-stress when the part is in actual use, shall be considerably less than the ultimate strength and well below the elastic limit. The ratio $\dfrac{\text{ultimate strength}}{\text{working unit-stress}}$ is known as the **factor of safety.**

Since there might be a factor of safety as far as strength is concerned and permanent deformation still occur, it is becoming more usual to base design on a comparison of the working unit-stress with the elastic limit. This gives what is often called the **real factor of safety.**

$$\text{Real factor of safety} = \frac{\text{elastic limit}}{\text{working unit-stress}}$$

The determination of the correct working unit-stress in any actual case and the selection of a proper factor of safety are matters requiring judgment and engineering experience. The particular

qualities of the timber, stone work, or steel to be used must be taken into account. The kind or kinds of stress to be provided for and the manner of application must be understood and allowed for, as suddenly applied loads produce higher unit-stresses than steady loads. Ample allowance must be made for possible overloads. In general, good design must be based upon the best engineering practice for the special demands to be met. In some cities the allowable working unit-stress is fixed by the building law. The following table of factors of safety is to be regarded as furnishing only general or *average* values.

FACTORS OF SAFETY

Material	For Steady Stress (Buildings, etc.)	For Varying Stress (Bridges, etc.)	For Repeated or Reversed Stresses (Machines)
Timber	8	10	15
Brick and stone	15	25	35
Cast iron	6	15	20
Wrought iron	4	6	10
Structural steel	4	6	10
Hard steel	5	8	15

PROBLEMS

15–23. A wrought-iron rod is to be under a stress of 90,000 lb. Compute its diameter for use in a building and also for use in a bridge truss.

15–24. A short wooden block, standing on end, is to support a load of 80,000 lb. If the block is 4×4 in. in section, what is the factor of safety?

15–25. What should be the section of a cast-iron block to support a steady load of 100,000 lb?

15–26. A structural steel tension member is to carry repeated stresses of 60,000 lb. What should be its cross section?

15–27. In laying the steel rails of a railroad, the rails were put down at a temperature of 50°. No allowance was made for expansion. What compressive unit-stress will be set up if the temperature of the rails increases to 85°? (Coefficient of expansion for steel, 0.0000065.)

15–28. What is the factor of safety if the working stress in a steel member is 12,000 lb, and the ultimate stress is 60,000 lb per sq in.?

15–29. What is the factor of safety if 9000 lb are supported by a steel rod $\frac{7}{8}$ in. in diameter, if the ultimate strength of steel is 60,000 lb?

15–30. What may be the load on a 2×3 in. wooden block, if the factor of safety is to be 8 and the ultimate strength is 5000 lb?

124. Stresses in beams. A bar or timber resting on supports at its ends constitutes a **simple beam.** Where a part of a beam projects out from a wall, or beyond a support, this part is called a **cantilever beam.**

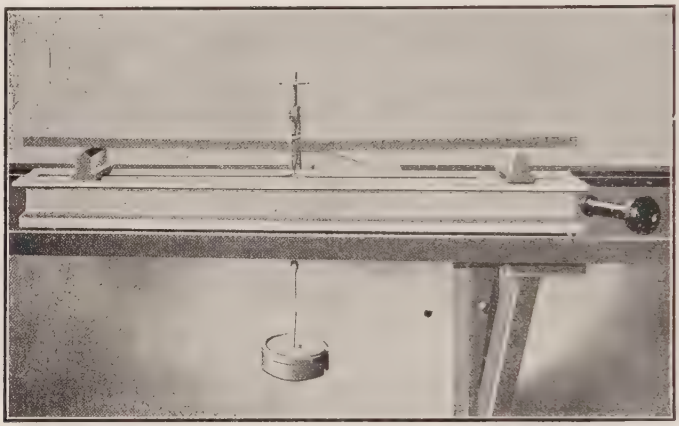


Fig. 220. Apparatus for illustrating the laws of bending.

The deflection of a simple beam under a central load may be studied by the laboratory device shown in Fig. 220, where the beam, represented by a rod of rectangular section, rests upon knife edges carried on a rigid support and the deflection for a given load is measured by determining with a micrometer screw the distance from a fixed level to the surface of the beam at the middle.

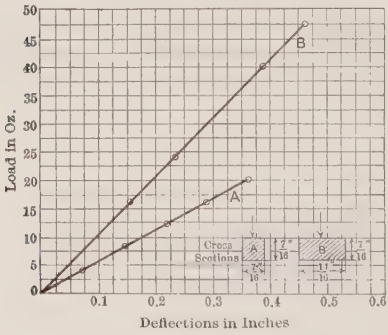


Fig. 221. Curves showing the relation of deflection to load with different widths of beam.

Figure 221, plotted from the data of such a test, shows that, when the elastic limit of the beam is not exceeded, the deflection is proportional to the load. The same apparatus may be used to show that, when not stressed beyond its elastic limit, the beam returns to its original position when the load is removed, and that, when stressed beyond its elastic limit, the beam takes a permanent set.

To arrive at an understanding of the kind and distribution of the stresses to which a beam under load is subjected, consider Fig. 222, which shows a simple beam with distributed load and with the deflection greatly exaggerated for the purposes of illustration.

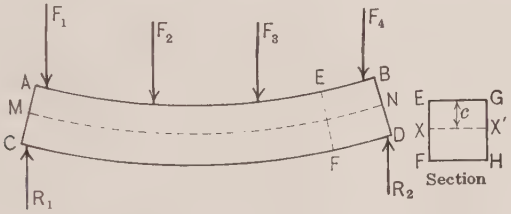


FIG. 222. Loaded simple beam, bending exaggerated.

It is apparent from the diagram that the edge AB of the beam is shortened by the bending, and that the edge CD is lengthened. The upper portion of the beam must therefore be under *compressive stress*, and the lower portion under *tensile stress*. At some place between AB and CD, as MN, there must be a line which, while curved because of the load, remains of the same length as when the beam was straight. A surface may therefore be imagined across the beam through MN, above which the fibres are under compression and below which they are under tension, but in which there is neither compression nor tension. This surface is known as the **neutral surface** for the beam. The line XX', which shows the position of this surface across a section of the beam, is called the **neutral axis of the section.**

The fibres of the beam are shortened most at AB; therefore at any section the compressive stress is zero at the neutral axis XX' and increases in proportion to the distance of the fibre from XX' to a maximum stress at EG. In the same way the tensile stress on the other side of the neutral axis is zero at XX' and increases with the distance to a maximum at FH. If a beam fails through compression, it will be most likely to yield first at the outer fibres EG, where the stress is greatest; and if it fails because the tension is too great, it will be most likely to yield at FH, where the tensile stress is a maximum. *In computing the safe load for a beam, therefore, it is necessary only to consider the stress at the extreme or outer fibres for any section.*

In a cantilever beam, as Fig. 223, there is, similarly, a neutral surface for the beam and a neutral axis for the section, but the

upper side AB of the beam is under tensile stress and the lower side is under compressive stress.

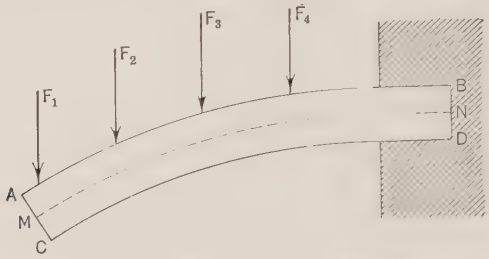


FIG. 223. Loaded cantilever beam, bending exaggerated.

125. Vertical shear in beams. In addition to the tensile and compressive stresses, there is a shearing stress at each section of a loaded beam. The vertical shear, across, or at right angles to the axis, is considered here, although in certain cases, such as wood members, the longitudinal shear, along the axis, is also important.

Consider, for example, the static conditions for a portion of a uniformly loaded beam, to the left of a section EF, represented as a free body in Fig. 224. The forces acting are the reaction R_1 at the left support, the equal loads F_1, F_2, and F_3 (which we will assume to include the weight of the beam), and the forces due to the internal stresses at the section. The horizontal stresses are the push C due to the fact that above the neutral surface the beam is in compression, and the pull T due to the fact that below the neutral surface the beam is in tension. C and T are distributed over the upper and lower half of the section, respectively. But from the assumption of the symmetrical loading, we know that $R_1 = R_2$ and therefore that R_1 *is less than* the sum $F_1 + F_2 + F_3$. To satisfy the condition that, for equilibrium, the sum of all the forces upward must equal the sum of all the forces downward, $\Sigma V = 0$, equation (2), there must be an upward vertical force V_1 supplied by the portion of the beam at the right of the section of such amount that

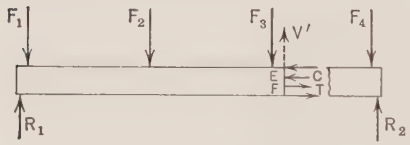


FIG. 224. Vertical shear in a beam.

$$R_1 + V_1 - F_1 - F_2 - F_3 = 0$$

In considering the whole beam, it will be seen that the upward

forces that balance F_1, F_2, F_3, and F_4 are R_1 and R_2 (the right reaction); V_1 is, therefore, an internal stress at the section. In other words, at the left of any section of a beam loaded as the one under consideration, there is a downward force V due to the loads and an upward reaction V_1 due to the material on the right of the section. V is therefore the **shearing stress** at the section. Beams must be designed to resist not only the tensile and compressive stresses but also any shearing stress at a section. In the beam just considered, the shearing stress is greatest at the supports where $V = R_1 = R_2$, and zero at the middle since $R_1 = R_2 =$ the load on each half of the beam. **The vertical shear at any section of a beam is the algebraic sum of all the vertical forces on the left of the section.**

Vertical shear diagrams, see Fig. 225, show the variation of the stress throughout the length of the beam. The vertical height represents to scale the amount of shear. If the load is distributed uniformly along the beam, the shear varies from a positive maximum at the left end to a negative maximum at the right, and crosses at zero at the center. Under a concentrated load the amount of shear varies abruptly from positive on the left of the load to negative on the right, being zero at the load. Figure 225 shows the diagrams for uniform loading, concentrated load at the middle point, and a combination of both of these.

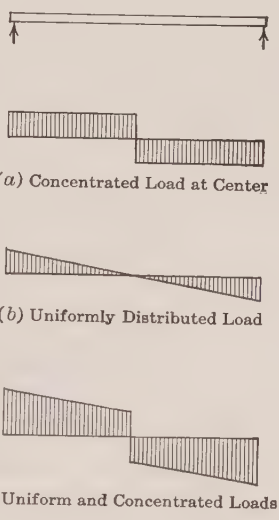


FIG. 225. Shear diagrams.

Note that the algebraic sign does not affect the value, but simply denotes direction, as in preceding chapters.

Example 1.

If a beam 16 ft long has a uniform load of 1000 lb, find the vertical shear (*a*) at the left support, (*b*) 4 ft from the left end, (*c*) at the center, (*d*) 4 ft from the right end, (*e*) at the right end.

Solution.

$$R_1 = R_2 = \frac{W}{2} = \frac{1000}{2} = 500$$

(*a*) $V = +R_1 - 0 = 500$ lb *Ans.* (There is no load *left* of R_1.)

(*b*) $V = +R_1 - \frac{W}{4} = 500 - 250 = 250$ b *Ans.* ($\frac{1}{4}$ load left of section.)

(*c*) $V = +R_1 - \frac{W}{2} = 500 - 500 = 0$ lb *Ans.* ($\frac{1}{2}$ load left of center.)

(*d*) $V = +R_1 - \frac{3W}{4} = 500 - 750 = -250$ lb *Ans.*

(*e*) $V = +R_1 - W = 500 - 1000 = -500$ lb *Ans.*

Example 2.

If a beam 16 ft long has a concentrated load of 1000 lb at the center, find its vertical shear, (*a*) at the left support, (*b*) 4 ft from the left end, (*c*) 4 ft from the right end, (*d*) at the right support, (*e*) at the center.

Solution.

(*a*) $V = +R_1 - 0 = 500 - 0 = 500$ lb *Ans.*
(*b*) $V = +R_1 - 0 = 500 - 0 = 500$ lb *Ans.*
(*c*) $V = +R_1 - P = 500 - 1000 = -500$ lb *Ans.*
(*d*) $V = +R_1 - P = 500 - 1000 = -500$ lb *Ans.*
(*e*) V at the center is just passing from $+500$ to -500. It equals zero under the concentrated load. *Ans.*

Note. The conventional method is to compute the shear and bending moment to the *left of the section.* It can be seen that the same amount would be obtained to the right of the section but with the opposite sign.

126. Bending moments. To determine the effect of the forces acting on a beam, and the stresses thereby set up, the reactions at the supports must first be found by the method of Chapter V.

The bending moment at any section of a beam is the algebraic sum of the moments of all the vertical forces at the left of that section. It is a measure of the tendency of these forces to produce rotation about an axis *at the section.*

Example 1.

Referring to the beam in Fig. 226, find the reactions, and the bending moment at sections under each load.

Solution. The reaction R_2 may be found from the equation of moments, $\Sigma M = 0$, equation (3), about the left support.

$$1500 \times 12 + 2000 \times 6 - R_2 \times 16 = 0$$

$$R_2 = 1875 \text{ lb} \quad \textit{Ans.}$$

Similarly, the reaction R_1 may be found from the equation of moments about the right support,

$$R_1 \times 16 - 1500 \times 4 - 2000 \times 10 = 0$$

$$R_1 = 1625 \text{ lb} \quad \textit{Ans.}$$

Check: $$R_1 + R_2 = 1500 + 2000$$

$$3500 = 3500$$

The bending moment at a support is zero. The bending moment at a section at the point of application of the 2000-lb load is

$$M = R_1 \times 6 = 1625 \times 6 = 9750 \text{ lb-ft} \quad \textit{Ans.}$$

The bending moment at a section at the point of application of the 1500-lb load is

$$M = 1625 \times 12 - 2000 \times 6 = 7500 \text{ lb-ft} \quad \textit{Ans.}$$

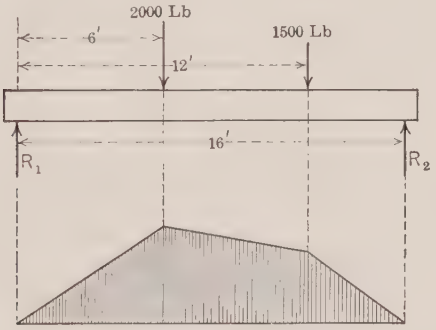


Fig. 226. Bending moments.

The cross-hatched area under the beam is the **bending-moment diagram** for the beam. In this diagram the vertical heights represent to scale the values of the bending moments for sections of the beam directly above. The diagram thus shows how the bending moments vary throughout the length of the beam. Maximum bending in this case occurs under the largest concentrated load; which is also the section of zero shear.

Example 2.

Figure 227 shows the bending moments for a 16-ft beam where the load, including the weight of the beam, is uniformly 200 lb per ft.

Solution. In this case the total load is $16 \times 200 = 3200$ lb, and $R_1 = R_2 = 1600$ lb.

If we consider a section 4 ft from the left support, the total load

and weight of the part of the beam to the left of this section is $4 \times 200 = 800$ lb. This may be considered as a single load applied at the middle of the 4-ft part, or 2 ft from the section. Therefore, for the section,

$$M = 1600 \times 4 - 800 \times 2 = 4800 \text{ lb-ft}$$

At a section at the middle of the beam,

$$M = 1600 \times 8 - 1600 \times 4 = 6400 \text{ lb-ft}$$

At a section 4 ft from the right support,

$$M = 1600 \times 12 - 2400 \times 6 = 4800 \text{ lb-ft}$$

In the bending-moment diagram, since the beam is uniformly loaded, the line joining the ends of the ordinates is a curve. It may be shown to be a parabola. Maximum bending occurs at the center.

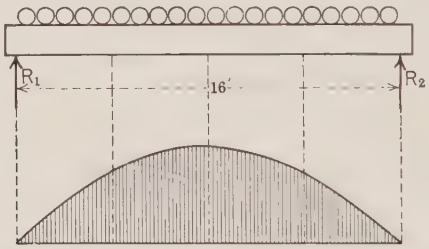


FIG. 227. Bending moments, beam uniformly loaded.

Example 3.

Figure 228 shows the bending-moment diagram for a beam which extends beyond a support,

Solution.

$R_1 = 1500$ lb, $R_2 = 3000$ lb

For a section at the 2000-lb load,

$$M = 1500 \times 4 = 6000 \text{ lb-ft}$$

For a section at the 1500-lb load,

$$M = 1500 \times 8 - 2000 \times 4 = 4000 \text{ lb-ft}$$

For a section at the right support,

$$M = 1500 \times 12 - 2000 \times 8 - 1500 \times 4 = -4000 \text{ lb-ft.}$$

A negative bending moment is one which causes the beam to bend with the upper edge convex. The same bending moment for a section at R_2 will be obtained by considering the projecting

end of the beam as a cantilever beam. The bending moment for a section at the support R_2, due to the load of 1000 lb on the outer end, is $1000 \times 4 = 4000$ lb-ft, and the load will obviously tend to bend the outer end of the beam downward so that the convex side of the beam is upward.

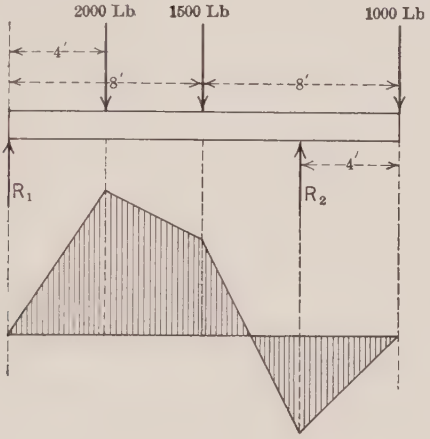


FIG. 228. Bending moments, part of beam overhanging.

If the load on a beam is concentrated at the center, then the left reacton $R_1 = F/2$, and the maximum bending moment around the section at the center, under the load, is

$$M = R_1 \times \frac{L}{2} = \frac{F}{2} \times \frac{L}{2} = \frac{FL}{4}, \text{ i.e.,}$$

$$M = \frac{FL}{4} \tag{56}$$

If the load is uniformly distributed, the left reaction $= W/2$. Since one-half the load is in this case on the left of the section,

The maximum moment $=$

$$M = \left(R_1 \times \frac{L}{2}\right) - \left(\frac{W}{2} \times \frac{L}{4}\right) = \left(\frac{W}{2} \times \frac{L}{2}\right) - \left(\frac{W}{2} \times \frac{L}{4}\right) = \left(\frac{WL}{4} - \frac{WL}{8}\right)$$

therefore,

$$M = \frac{WL}{8} \tag{57}$$

Note that the arm of the half-load on the left is from the center of gravity of that half to the center $= \frac{1}{4}$ the length.

Since $M = FL/4$ if the load is concentrated at the center and $= WL/8$ if it is uniformly distributed, it can be seen that a beam will be twice as strong if a load is distributed uniformly rather than concentrated.

It can be shown also that for a cantilever beam with concentrated load at the end

$$M = FL \tag{58}$$

and for a cantilever under uniform load

$$M = \frac{WL}{2} \tag{59}$$

127. Resisting moment. Referring again to Fig. 224, it will be seen that the bending moment for the section EF tends to turn the beam in a clockwise direction about the neutral axis of the section. To satisfy the second condition of equilibrium, therefore, the moments of the distributed stresses C and T acting on the section at EF must supply an equal tendency to turn the portion of the beam counter-clockwise about the same axis. **The algebraic sum of the moments of all the horizontal stresses at a section, with respect to the neutral axis of the section, is called the resisting moment.** Hence at any section,

The bending moment = the resisting moment

128. General equation for stress. From the statements of Art. 126 and Art. 127, it is possible to derive a general expression for the horizontal unit-stress at the top or bottom of a beam, at any given section.

Let s be the horizontal unit-stress at the upper or lower side of the beam, c the distance from the neutral axis to upper or lower side of the beam; then, since within the elastic limit the horizontal stresses in any section increase uniformly from the neutral axis to the top and bottom of the beam, the horizontal unit-stress at any distance from the neutral axis less than c, as r, is $s \times r/c$. On any small elementary area a of the section at the distance r from the neutral axis, the total stress is therefore $as \times r/c$ or $s/c \times ar$. The moment of this stress with respect to the neutral axis is therefore $s/c \times ar$ multiplied by its moment arm r, or

$s/c \times ar^2$. And since, by definition, the resisting moment is the algebraic sum of all these elementary moments for all possible values of r,

$$\text{Resisting moment} = \frac{s}{c} a_1 r_1^2 + \frac{s}{c} a_2 r_2^2 + \frac{s}{c} a_3 r_3^2 \ldots + \frac{s}{c} a_n r_n^2 = \frac{s}{c} (\Sigma\, ar^2)$$

The quantity Σar^2 is called the **moment of inertia of the section** of the beam, see following article, and is represented by I; hence we may write the above equation in the form

$$\text{Resisting moment} = \frac{sI}{c} \tag{60}$$

In Art. 127 it was shown that the bending moment for any section is equal to the resisting moment for that section, therefore,

$$M = \frac{sI}{c} \quad \text{and} \quad s = \frac{Mc}{I} \tag{61}$$

The above equations hold true for all kinds of beams. The second is used in computing the unit-stress s at the outer fibres of any section, where the stress is greatest. The quantity I/c is called the **section modulus** and depends upon the shape of the cross section.

129. Moment of inertia. The moment of inertia of a sectional area with respect to an axis is the sum of the products obtained by multiplying each elementary area in the section by the square of its distance from that axis. If a represents a small elementary area, r its distance from the axis, and I the moment of inertia of the section, then:

$$I = \Sigma ar^2 \tag{62}$$

This equation gives I with relation to an axis through the center of gravity. If it is desired to obtain the moment of inertia about a parallel axis at a distance d from the center of gravity, it may be obtained by the **transfer equation**:

$$I = I_0 + Ad^2 \tag{63}$$

in which I_0 is the moment of inertia of an area with respect to an axis through its center of gravity and I is the moment of

inertia with respect to any other axis parallel to it and at a distance d from it.

Thus, for example, the moment of inertia of a rectangular section about its neutral axis (through the center of gravity) is $bh^3/12$ (see table below). The moment of inertia about an axis at the *base* of the rectangle can be found by equation (63). Here the distance $d = \frac{1}{2}$ the height h.

$$I = \frac{bh^3}{12} + Ad^2 = \frac{bh^3}{12} + bh \times \left(\frac{h}{2}\right)^2 = \frac{bh^3}{12} + \frac{bh^3}{4} = \frac{4\,bh^3}{12} = \frac{bh^3}{3}$$

The solution of the equation $\sum ar^2$ involves the use of higher mathematics and hence will not be attempted here. The values of I for some of the more commonly used shapes are listed below. The values of I for structural shapes are given in handbooks.

Rectangle	$\frac{bh^3}{12}$
Triangle	$\frac{bh^3}{36}$
Circle	$0.0491\ D^4$
Semi-circle	$0.1098\ r^4$
Hollow rectangle	$\frac{b_1h_1^3 - b_2h_2^3}{12}$
Hollow circle	$0.0491\ (D_1^4 - D_2^4)$

The equations for the **moment of inertia of an area** and the **moment of inertia of a body** are similar in form. The moment of inertia of a body is defined as the sum of the products obtained by multiplying the *mass* of each particle of the body by the square of its distance from the axis of rotation. Letting m represent the mass of each particle and r its distance from the axis, then:

$$I = \sum mr^2 \text{ (as in Art. 77)} \qquad (23)$$

130. The location of the neutral axis. In the consideration of a portion of a beam, Fig. 224, as a body in equilibrium, the equation $\sum V = 0$ was found to lead to an expression for the shear at a section. Similarly, the equation $\sum M = 0$ leads to the general formula for stress, $s = Mc/I$.

It is necessary also that the third equation for equilibrium, $\Sigma H = 0$, be true. The horizontal stresses are the tensile stresses at the section EF in one direction, and the compressive stresses in the opposite direction. The summation of the stresses on all the elements of area for the section is $(s/c)\,\Sigma ar$. For an axis within the section some of these radii, r, are negative, and when the axis passes through the center of gravity of the section, $\Sigma ar = 0$. Therefore $(s/c)\,\Sigma ar = 0$, or in other words *the algebraic sum of all the stresses is zero, when the neutral axis is through the center of gravity of the section.* This is a reason why it is important to be able to find the center of gravity of an area (see Art. 41). By finding the center of gravity of the section of a beam we determine the position of the neutral axis.

Example 1.

A beam 10 ft long, 4 × 10 in. in section, resting on edge, is supported at the ends and bears a distributed load of 200 lb per ft. What will be the maximum stress and the shearing force?

Solution. Maximum bending moment is at the middle, and is

$$\frac{WL}{8} = \frac{1}{8} \times 200 \times 10 \times 120 = 30{,}000 \text{ in-lb}$$

$$I = \frac{bd^3}{12} = \frac{4 \times 10 \times 10 \times 10}{12} = \frac{1000}{3}$$

Therefore,

$$\text{Maximum stress } s = \frac{Mc}{I} = \frac{30{,}000}{\frac{1000}{3}} \times 5 = 450 \text{ lb per sq in.} \quad \textit{Ans.}$$

Maximum shearing force is at the ends, and is

$$V = \frac{1}{2} W = \frac{200 \times 10}{2} = 1000 \text{ lb} \quad \textit{Ans.}$$

Example 2.

A round steel rod 18 in. long in the clear, and $1\frac{1}{2}$-in. diameter, is fixed at one end. What load may be suspended from the other end if the maximum allowable stress is 5000 lb per sq in.?

Solution. Maximum bending moment is at end which is fixed and $= W \times 18$ in-lb.

$$I = \frac{\pi D}{64} = \frac{891}{3584}$$

Therefore,

$$s = \frac{Mc}{I} \quad 5000 = \frac{\frac{W \times 18}{891}}{3584} \times \frac{3}{4}, \text{ from which } W = 92.1 \text{ lb} \quad Ans.$$

131. Strength of beams. It has been shown that for a rectangular section, $I = bh^3/12$ and $c = h/2$. Also for a concentrated load at the center of a beam, $M = FL/4$. Since $M = sI/c$, the above values can be substituted, giving $FL/4 = sbh^3/12 \div h/2 = sbh^2/6$ from which $F = 4sbh^2/6L$. Since $\frac{4}{6}$ is constant in value, and s can be considered constant for any material,

$$F \propto \frac{bh^2}{L} \tag{64}$$

that is, the **strength of a rectangular section varies directly as the breadth of the section, directly as the square of the depth, and inversely as the length.**

For other shapes, the strength varies as the section modulus (I/c) and inversely as the length.

132. Stiffness of beams. The stiffness of a beam depends upon the load and its distribution, upon the modulus of elasticity, and upon the moment of inertia of the section. The deflection of a beam (represented by y) can be determined experimentally, or can be computed by formulas which have been derived for different loadings. Some of these are:

For cantilever beam with concentrated load at end $$y = \frac{FL^3}{3EI} \tag{65}$$

For cantilever beam with uniform load $$y = \frac{WL^3}{8EI} \tag{66}$$

For simple beam with concentrated load at center $$y = \frac{FL^3}{48EI} \tag{67}$$

For simple beam with uniform load $$y = \frac{5WL^3}{384EI} \tag{68}$$

A consideration of these equations shows that the deflection y varies directly as the cube of the length, and inversely as the moment of inertia, assuming that the loading and material are the same. Since **the stiffness of a beam varies inversely as the**

deflection (i.e., a stiff beam will deflect less than one not as stiff), and since $I = bh^3/12$, it follows that:

$$\text{Stiffness} \propto \frac{bh^3}{L^3} \tag{69}$$

That is, **the stiffness of a rectangular section varies directly as the breadth of the section, directly as the cube of the depth, and inversely as the cube of the length.**

Example.

Compare the strength and stiffness of a 2 by 6 in. plank used as a beam, first on edge and then flat.

Solution.

Strength varies as bh^2/L therefore the second position is

$\frac{6}{2} \times \frac{2}{6} \times \frac{2}{6} \div \frac{1}{1} = \frac{1}{3}$ as strong as the first *Ans.*

Stiffness varies as bh^3/L^3, therefore the second position is

$\frac{6}{2} \times \frac{2}{6} \times \frac{2}{6} \times \frac{2}{6} \div 1^3 = \frac{4}{36} = \frac{1}{9}$ as stiff *Ans.*

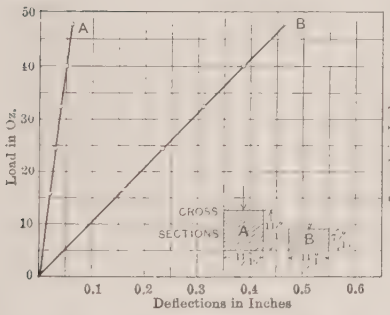


FIG. 229. Curves showing the relation of deflection to depth of beam.

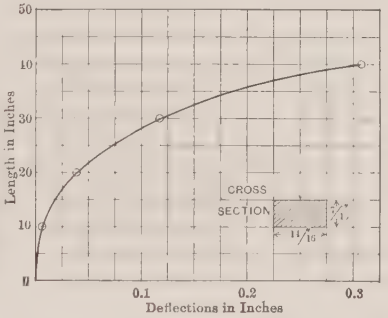


FIG. 230. Curve showing the relation of deflection to length of beam.

133. Columns. Columns are members supporting compression lengthwise. Unless they are short struts (less than eight or ten times as long as their least dimension), or are side braced against flexure, they must resist bending stresses in addition to compression. In case the load is not applied centrally on the end of the column, that is, if the loading is eccentric, there will be additional bending stresses to be carried.

The tendency toward bending increases as the length increases. The resistance to bending varies as the breadth and depth of the

section. The farther the mass is distributed from the neutral axis, the less will be the tendency to bend. Since the radius of gyration k is the distance from the axis to the point where the mass can be considered to act, bending varies inversely as k.

Several formulas have been developed, which combine a term for compression with a term for flexure, the latter term usually involving the *slenderness ratio* $= L/k$. **Rankine's formula** is one of the oldest column formulas and uses the square of the slenderness ratio. The formula may be stated:

$$s = \frac{F}{A}\left(1 + \frac{qL^2}{k^2}\right) \qquad (70)$$

in which q is a constant depending upon the material and the arrangement of the ends, whether *fixed*, or *round*, etc. The constants are listed in reference books, some of them being: for timber with both ends fixed $q = 1/3000$; for cast iron $q = 1/5000$; for steel $q = 1/25{,}000$.

Several *straight line* formulas are in common use, some using the first power instead of the square of the slenderness ratio, and some using the ratio of L to d, the least dimension, instead of L to k.

The straight line formula usually takes the form:

$$\frac{F}{A} = s - C\frac{L}{k} \qquad (71)$$

in which C is an experimental coefficient depending upon the material and the arrangement of the ends of the column.

There is a diversity of values for C and s, and a large number of formulas are in use. The following are typical.

$$\text{For steel columns } \frac{F}{A} = 18{,}000 - 70\frac{L}{k} \qquad (72)$$

with a recommended maximum of 14,000, and a slenderness ratio not over 120.

$$\text{For cast-iron columns } \frac{F}{A} = 9000 - 40\frac{L}{k} \qquad (73)$$

with a maximum L/k ratio of 90.

$$\text{For wooden columns } \frac{F}{A} = s\left[1 - \frac{1}{3}\left(\frac{L}{Kd}\right)^4\right] \qquad (74)$$

in which $K = \left(\frac{\pi}{2}\right)\sqrt{\frac{E}{6\,s}}$, d is the least dimension, used instead of k, s is the stress along the grain, and E is the modulus of elasticity. This formula, recommended by the ASTM,* is later than one very commonly used over a period of years known as the AREA† wooden column formula:

$$\frac{F}{A} = s\left(1 - \frac{L}{60\,d}\right) \tag{75}$$

The foregoing equations will be satisfactory for most column problems. For actual design, since there is a difference of opinion or of practice as to the values to be used, a modern handbook or recommended code should be consulted.

134. Sections of beams and columns. Short columns or columns prevented from bending, may be expected to yield only at a stress equal to the ultimate compressive strength of the material. Long columns not supported laterally yield by bending. It is desirable, therefore, to shape the section of posts and struts to offer the greatest possible resistance to bending. This is accomplished, as is seen from the laws for stiffness just stated, when the material is distributed as widely as possible about the axis of the column.

In simple beams the top must resist compressive stress and the bottom tensile stress. The bottom flanges may therefore be thin plates as a pull does not produce bending, but the top must be shaped to resist the tendency to bend under compression.

Cast iron has a compressive strength nearly four and one-half times as great as its tensile strength. The section of cast-iron girders is, therefore, usually made so that the area under tension is approximately four and a half times as large as that under compression. Figure 231(*a*) shows some common sections for cast-iron girders.

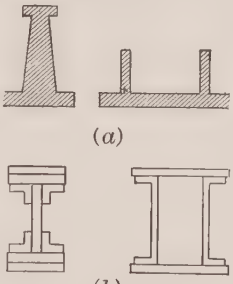


FIG. 231. Sections of girders.

Steel and wrought iron have practically the same ultimate strength for both tension and compression. Symmetrical and

* American Society for Testing Materials.

† American Railway Engineering Association.

economical " built-up " sections similar to those of Fig. 231(*b*) are therefore common.

In general the function of the flanges is to resist bending moment, that of the web to resist the shearing force. Any web strong enough to hold the flanges in position and to permit the girder to be handled is usually amply sufficient to withstand the shearing force.

In beams of uniform section, only the maximum bending moment need be considered. The section must be designed to sustain this safely. Girders of " equal strength " may be designed in which the section varies according to variation in bending moment.

PROBLEMS

15–31. A steel beam 10 ft long has a load of 3000 lb at the center. The value for I/c is 7.9. Find the fibre stress.

15–32. A 4 × 6 in. beam of yellow pine has a span of 8 ft. Using 1300 lb as safe stress, find the load it can carry at the center.

15–33. A steel bar, 1 in. × 2 in. × 4 ft long, is to be used as a beam supporting a load of 200 lb at the middle. Find the stress when used on edge and when used flat.

15–34. A beam 10 ft long and 4 by 10 in. in section, resting on the 4-in. face, is supported at the ends and bears a uniformly distributed load of 400 lb per ft. Determine the bending moments and draw to scale the bending-moment diagram. Compute also the maximum shear.

15–35. A round steel rod 18 in. long in the clear and $1\frac{1}{2}$ in. in diameter is fixed at one end. What load may be suspended from the other end if the maximum unit-stress allowable is 5000 lb per sq in.?

15–36. What will be the greatest stress in a bar 2 sq in. in sectional area for a bending moment of 20,000 in-lb, if (*a*) round? (*b*) square?

15–37. A beam AB, 16 ft long, weighing 200 lb per ft, is supported at both ends. It bears loads as follows: 5 tons 4 ft from the end A, 10 tons at the middle, 6 tons 5 ft from the end B. Compute the bending moments due to the loads and weight of the beam combined for sections 2 ft, 4 ft, 8 ft, and 10 ft from A.

15–38. The top and bottom flanges of a girder are each 6 × $\frac{5}{8}$ in. The web is $\frac{5}{8}$ in. thick and the depth of the beam over all is 10 in. What bending moment will be required to give a maximum tensile stress of 10,000 lb per sq in.?

15–39. Locate the neutral axis for the section shown in Fig. 89(*c*).

15–40. A wooden beam 6 × 8 in. in section and having a span of 12 ft rests on a 6-in. face. It carries a uniformly distributed load of 6000 lb. Compute the maximum unit fibre-stress and the factor of safety.

15–41. A simple beam of 16-ft span 6 × 8 in. in section carries a uniform load, including its weight, of 60 lb per linear ft. It carries a concentrated

load of 600 lb at a distance of 4 ft from the left end. Compute the bending moments for three selected sections of the beam and construct the bending-moment diagram. What and where is the maximum unit fibre-stress if resting on a 6-in. face?

15–42. A simple beam weighing 50 lb per linear ft is 10 ft in span and has a load of 500 lb at the middle. Compute the maximum bending moment, and the maximum vertical shear.

15–43. A floor is supported by 3 × 8-in. wooden joists of 16-ft span, spaced 18 in. center to center. When the floor carries a total load of 250 lb per sq ft, what is the factor of safety of the joists?

135. Torsion. If we mark parallel straight lines from end to end upon the surface of a rod of elastic material, and then clamp one end of the rod fast and apply to the other end a couple which tends to rotate this end about the axis of the rod, we find that the twist is communicated along the entire rod to the end which is fast in such a manner that the straight lines become helixes about the rod. The actual displacement from its original position of a point in any line varies directly with its distance from the fixed end, being zero at this end and a maximum at the free end where

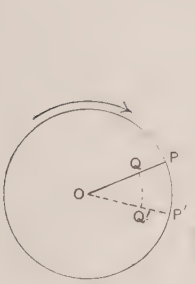


Fig. 232.

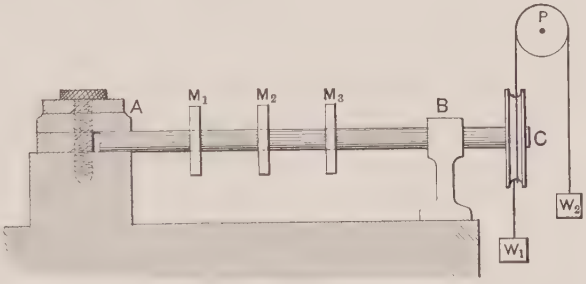


Fig. 233. Apparatus for studying torsion.

the torque is applied. This is the general phenomenon known as *twisting* or *torsion*. A little consideration will make it clear that **the stress imparted to the material of the rod is a kind of shearing.** The fibres have been displaced through an angle with the axis of the rod, and the particles at every section have been rotated through an arc the center of which lies at the center of the section. The outer particles at a section as P, Fig. 232, have moved to the new position P' a particle nearer the axis, as Q, has moved to the new position Q', while particles at the center O have not been rotated at all. The shearing stress at a section is greater and

greater, as the material lies farther from the axis O; it is also greater the farther the section is taken from the fixed end of the rod.

136. Experimental study of torsion. A simple laboratory apparatus for the study of torsion is shown in Fig. 233. A clamp, which may be adjusted to rods of different diameter, holds the end A of the rod to be twisted; the other end rests in a loose bearing B. Torque is applied by means of cords passed around the grooved wheel C and supporting the equal weights W_1 and W_2. In this way pressure upon the bearing B is avoided. Adjustable mirrors, M_1, M_2, M_3, etc., are attached to the rod by sliding spring sleeves, at the equal intervals AM_1, M_1M_2, M_2M_3, etc., and the amount of twist for each length is determined by viewing in these mirrors, by means of a telescope, the image of a vertical scale placed at relatively so great a distance that it is practically equally distant from each mirror. The twist may be taken as proportional to the reading in scale divisions for each mirror, or the actual angle of twist may be found from the relation:

$$\frac{\text{Change in scale reading}}{\text{Distance scale to mirror} \times 2} = \text{tan. angle of twist}$$

With this apparatus it may be shown experimentally that:

1. If a twisting moment T produces an angle of twist ϕ, a moment of $2\,T$ will produce a twist of $2\,\phi$, etc. Or, the *angle of twist is proportional to the twisting moment.*

2. If the distances $AM_1 = M_1M_2 = M_2M_3$, etc., and the twist at M_1 is 10 scale divisions, the twist at M_2 will be 20 divisions, at M_3 30 divisions, etc. Or, the *angle of twist for the same torque is proportional to the length of the rod.*

3. For round rods or shafts of different diameters, *the angle of twist for the same torque varies inversely as the fourth power of the diameter.*

4. For rods of the same size but different materials, the twist for the same twisting moment varies inversely as a quantity known as the *modulus of rigidity* of the rod.

Where not one but several separate twisting moments are applied to the rod, the total twisting moment is the algebraic sum of the separate moments. If these separate twisting moments are applied at different points, as, for example, in a shaft carrying several pulleys on which belts are acting, the twisting moments

are different on the different sections between pulleys and may be found from the data by considering each section separately.

137. Formula for the torsion of shafts. When a twisting moment is applied to a shaft, each cross section of the shaft tends to twist with respect to the cross section next to it. Shearing stresses exist in all parts of the section. These stresses are zero at the center of the section and greatest at the boundary, and they act everywhere perpendicular to the moment arms drawn to them from the center. Within the elastic limit for the material of the shaft, the stresses are proportional to their moment arms. By a method similar to that of Art. 128, it may be shown that the fundamental equations for the torsion of shafts are:

$$T = \frac{sJ}{c} \quad \text{and} \quad s = \frac{Tc}{J} \tag{76}$$

in which T is the twisting moment or torque, s is the unit-stress (here always a shearing stress), c the distance from the center of the section to the outer fibres, i.e., the radius of the shaft, and J the polar moment of inertia for the section. The **polar moment of inertia** of a surface is the sum of the products obtained by multiplying each elementary area by the square of its distance from the center of the surface. The polar moment of inertia of a solid circle, as for example the cross section of a round shaft, is $\pi D^4/32$, where D is the diameter of the circle.

It should be noted that shafting which transmits power is subjected to variable loads, and that in consequence the permissible value of s is low.

138. Shafts transmitting power. Power from the driving unit is usually transmitted to a shaft by belts, and the shaft then transmits the power to machines or tools where the desired work is to be performed. F is the tangential driving force in pounds from the belt, r the radius of the pulley in inches, and n the number of revolutions per minute of the shaft and pulley, then in one revolution a force of F pounds acts through a distance of $2\,\pi r$ inches, and the work of $F \times 2\,\pi r$ inch-pounds, or $\pi\, Fr/6$ foot-pounds is performed. In one minute $n\pi Fr/6$ foot-pounds of work are performed. Therefore

$$\text{Horsepower} = \frac{n\pi Fr}{6 \times 33{,}000} \tag{77}$$

If the twisting moment Fr in this equation is replaced by the resisting moment sJ/c, the equation becomes

$$\text{Horsepower} = \frac{n\pi sJ}{6 \times 33{,}000 \times c} \tag{78a}$$

or

$$\frac{sJ}{c} = \frac{6 \times 33{,}000 \times P}{n\pi} \tag{78b}$$

in which P represents the horsepower.

It should be noted that, for a given value of P, s varies inversely with n, or in other words, for a given power transmitted to a shaft the slower the speed the greater the stress in the shaft.

PROBLEMS

15–44. A pulley 30 in. in diameter is carried on a 2-in. shaft. If the effective pull of the belt on the pulley rim is 500 lb, what is the unit-stress? If the shaft is steel, what is the factor of safety of the shaft?

15–45. A 3-in. line shaft runs at 300 rpm. What horsepower can be transmitted with a unit fibre-stress of 6000 lb per sq in.?

SUMMARY

Elasticity may be defined as the property of bodies by virtue of which they resume their original form when the external forces acting on them are removed.

Stress is the internal resistance which balances an external force.

Strength is the unit-stress which a material can sustain without failure.

Unit-stress equals the stress per square inch.

$$s = \frac{F}{A} \tag{53}$$

Hooke's Law is that a body acted on by an exterior force has a **deformation,** or change in shape, proportional to the stress, as long as the elastic limit of the material is not exceeded.

The unit-deformation is the deformation per unit of original length.

The modulus of elasticity is the ratio of unit-stress to unit-deformation.

$$E = \frac{\frac{F}{A}}{\frac{e}{L}} = \frac{FL}{Ae} \tag{54}$$

The stress due to temperature is given by the equation:

$$s = CtE \tag{55}$$

The factor of safety is the ratio of the ultimate strength to the working unit-stress. The real factor of safety, so called, is the ratio of the elastic limit to the working unit-stress.

Bending in beams causes compression in the fibres on one side of the neutral axis, and tension on the other. The amount varies from zero at the neutral axis to maximum at the extreme edge. The **neutral axis** passes through the center of gravity of the cross section of the beam.

Shear in beams occurs both longitudinally (horizontally) and vertically. The **vertical shear** at any section equals the algebraic sum of all the vertical forces on the left of the section.

The bending moment at any section equals the algebraic sum of the moments of all the vertical forces at the left of the section. The maximum moment usually occurs under the largest concentrated load, which is also the section of zero shear.

For a concentrated load at the center of a simple beam $$M = \frac{FL}{4} \tag{56}$$

For a uniform load on a simple beam $$M = \frac{WL}{8} \tag{57}$$

For a concentrated load at the end of a cantilever beam $$M = FL \tag{58}$$

For a uniform load on a cantilever beam $$M = \frac{WL}{2} \tag{59}$$

The resisting moment at any section equals the algebraic sum of the moments of all the horizontal stresses at the section with respect to the neutral axis as the center of rotation. **The bending moment equals the resisting moment.**

$$\text{Resisting moment} = \frac{sI}{c} \tag{60}$$

$$M = \frac{sI}{c} \quad \text{and} \quad s = \frac{Mc}{I} \tag{61}$$

The moment of inertia of a sectional area with respect to an axis is the sum of the products obtained by multiplying each elementary area in the section by the square of its distance from that axis.

$$I = \Sigma ar^2 \tag{62}$$

$$I = I_0 + Ad^2 \tag{63}$$

For a body $$I = \Sigma mr^2 \tag{23}$$

The values of I for some areas are given in Art. 129.

The strength of a rectangular beam varies directly as the breadth of the section, directly as the square of the depth, and inversely as the length. In general, the strength varies as the section modulus (I/c) and inversely as the length.

$$F \propto \frac{bh^2}{L} \tag{64}$$

The stiffness of a rectangular beam varies directly as the breadth of the section, directly as the cube of the depth, and inversely as the cube of the length. The stiffness varies inversely as the deflection y.

For a cantilever beam with concentrated load at the end $$y = \frac{FL^3}{3\,EI} \tag{65}$$

For a cantilever with uniform load $$y = \frac{WL^3}{8\,EI} \tag{66}$$

For a simple beam with concentrated load at the center $$y = \frac{FL^3}{48\,EI} \tag{67}$$

For a simple beam with uniform load $$y = \frac{5\,WL^3}{384\,EI} \tag{68}$$

$$\text{Stiffness} \propto \frac{bh^3}{L^3} \tag{69}$$

Columns are members supporting compression lengthwise. They are subject to bending as well as compression. Most column formulas include a term based on the **slenderness ratio** $= L/k$.

Rankine's formula is $$s = \frac{F}{A}\left(1 + \frac{qL^2}{k^2}\right) \tag{70}$$

The straight line formula is $$\frac{F}{A} = s - C\frac{L}{k} \tag{71}$$

For steel columns $$\frac{F}{A} = 18{,}000 - 70\frac{L}{k} \tag{72}$$

For cast-iron columns $$\frac{F}{A} = 9000 - 40\frac{L}{k} \tag{73}$$

For wooden columns (ASTM) $$\frac{F}{A} = s\left[1 - \frac{1}{3}\left(\frac{L}{Kd}\right)^4\right] \tag{74}$$

For wooden columns (AREA) $$\frac{F}{A} = s\left(1 - \frac{L}{60\,d}\right) \tag{75}$$

Torsion is a twisting stress which is a form of shear.

$$T = \frac{sJ}{c} \quad \text{and} \quad s = \frac{Tc}{J} \tag{76}$$

Horsepower in shafts depends on the torque.

$$\text{Horsepower} = \frac{n\pi Fr}{6 \times 33{,}000} \tag{77}$$

The relation between the resisting moment and horsepower is:

$$\text{Horsepower} = \frac{n\pi sJ}{6 \times 33{,}000 \times c} \tag{78a}$$

or

$$\frac{sJ}{c} = \frac{6 \times 33{,}000 \times P}{n\pi} \tag{78b}$$

CHAPTER XVI

MECHANICS OF FLUIDS

Chapter XVI treats of the fundamental laws of liquids and gases. The importance in mechanics of the action of fluids is readily apparent. The applications include hydraulic machinery of many kinds, hydraulic jacks, shock absorbers, brakes, circuit breakers, pumps, machines for testing strength of materials, etc. The principles of fluids apply to ships in the water and planes in the air. They apply also to the use of water and steam in the power plant, and of refrigerant liquids and gases in cooling systems; and to the air and its moisture content in heating, ventilating, and other air-conditioning operations.

139. Fluids. Matter may exist in the form of solids, as iron, wood, ice, etc., or in the form of fluids, as water, oil, steam, oxygen, air, etc.

The distinguishing property of a solid is that it has a definite shape and volume which cannot be changed without the application of an external force. Thus a piece of iron has a particular shape and an unchanging volume wherever it is placed. A fluid, on the other hand, has no definite shape, but molds itself to the shape of the vessel that contains it. Thus we may have a given quantity of water or of air in a vessel and may transfer it to another vessel to whose shape it will conform itself. Also a solid cannot be stirred or cut without the application of considerable force, but a fluid offers very little resistance to the disturbing force. These examples show that a fluid is composed of particles that are *mobile*, i.e., they move freely among each other and can easily be separated.

Furthermore, if we take a straight cylindrical tube open at one end and closed at the other, and fit into it a smooth rod of some solid substance such as iron or wood, and press the end of the rod in the direction of its length, the pressure applied is transmitted to the closed end of the tube without producing any effect on the curved surface; but if the tube contains a fluid, either a liquid or a gas, and pressure be applied to it by means of a tight

piston fitting into the tube, pressure will be felt not only on the base of the vessel, but also at all points of the curved surface. Thus fluid bodies are those that cannot sustain a longitudinal pressure, however small, without being supported by a lateral pressure also.

140. Distinction between liquids and gases. Fluids may be either liquids or gases. Liquids have a definite volume but no definite shape, whereas gases have *neither* a definite volume nor a definite shape. Thus water is, like all true liquids, practically incompressible, but will adapt itself to the shape of whatever vessel it is put into. But gas, like air or illuminating gas, has no definite shape and may be expanded by lessening the pressure upon it or compressed to any extent by increasing the pressure. These properties of solids, liquids, and gases may also be expressed as follows:

Solids have elasticity of *both* shape and volume.

Liquids have no elasticity of shape, but perfect elasticity of *volume.*

Gases have elasticity of *neither* shape nor volume.

It should be noticed that the distinctions above are of a general nature only; there are some substances that exist in an intermediate state, like coal tar, thick glue, etc. Also there are **vapors** which can be made to pass by imperceptible degrees from the gaseous to the liquid state. There are no strict lines of division between the three states of matter. The same substance may, under differing conditions of temperature, pressure, etc., exist in any one of the three states, as steam, water, and ice. Iron which is, as we know it, either a solid or a liquid, exists in the sun only as a gas.

141. Density. Iron is *more dense* than wood and water is *more dense* than air. By these statements it is meant that a given volume of one substance has more weight (or mass) than the same volume of the other.

The density of a substance is stated by giving the *weight of a unit volume of it.* Thus the density of cast iron is 0.26 lb per cu in., and also 450 lb per cu ft; the density of water is 62.5 lb per cu ft, and 0.0361 lb per cu in., etc. **Density equals weight divided by volume.** Letting density be represented by ρ (rho), weight by W, and volume by V, the equation is:

$$\rho = \frac{W}{V} \tag{79}$$

A knowledge of the density of the common materials of construction is very valuable to designers and builders because it enables them to compute the approximate weight of any particular part of a machine or structure. The volume of a part determined from the scale drawing multiplied by the density equals the weight of the finished part. The densities of a few of the more common materials of construction are given in the Appendix.

142. Specific gravity. By the *specific gravity* of a substance we mean its relative density as compared to water; or, in other words, *how many times heavier a particular body is than an equal volume of water.** Thus we say the specific gravity of lead is 11.35, meaning that lead is 11.35 times as heavy as water. A cubic foot of lead, therefore, weighs approximately $11.35 \times 62.5 = 710$ lb, and a cubic inch weighs $11.35 \times 0.0361 = 0.41$ lb.

$$\text{Specific gravity} = \frac{\text{weight of the substance}}{\text{weight of an equal volume of water}} = \frac{\text{density of substance}}{\text{density of water}}$$

In the metric units, the *density* of a body and its *specific gravity* are expressed by the same number. Thus the density of lead is 11.35 grams per cubic centimeter, and its specific gravity is 11.35.

An experimental method of determining the specific gravity of a *liquid* is by means of an open arm manometer, or glass U-tube; see Fig. 234. The procedure is to pour some mercury into the tube until it rises an inch or so in each arm. Since it is a liquid, both arms will be at the same level. Then fill one arm about two-thirds full of water, which will cause the mercury to rise somewhat in the other arm to balance the weight of the water. Since the specific gravity of mercury is 13.6, the water column will stand 13.6 times as high as the mercury column (the difference between the mercury levels).

Then pour some other liquid, as turpentine, into the other arm until the mercury has returned to its original position, at the same

* Strictly, the comparison should be to an equal volume of *pure water at* 4 C, the temperature at which water is most dense. The density of water changes so slightly with change of temperature, however, that for practical purposes the comparison may be made at any convenient temperature.

level in both arms. Since the density of turpentine is less than that of water, it will take more volume to balance the water, and the turpentine will stand higher in the tube. The weight of the water is now balanced by the weight of the turpentine, and their densities are inversely proportional to their volumes. The volumes are proportional to the lengths (heights) of the columns.

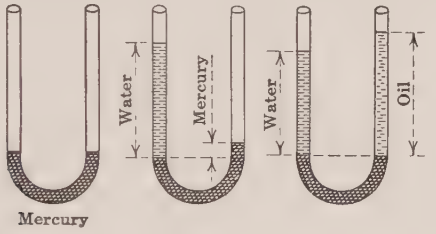


FIG. 234.

Suppose that the water column measures 8.7 in. and the turpentine exactly 10 in.; then the density of turpentine is to the density of water as the height of the water is to the height of the turpentine, $= 8.7 \div 10.0 = 0.87$, which is the specific gravity of turpentine.

THE SPECIFIC GRAVITY OF SOME LIQUIDS

Alcohol	0.80	Linseed oil	0.93
Fuel oil	0.85	Turpentine	0.87
Gasoline	0.69	Water, fresh	1.000
Kerosene	0.79	Water, salt	1.026
		Mercury	13.6

PROBLEMS

(Consult tables in Appendix for required density or specific gravity)

16–1. What is the weight of a 12-ft length of steel shafting of $1\frac{1}{8}$-in. diameter?

16–2. Find the size of a round iron shot to weigh 16 lb.

16–3. Find the weight of a column of mercury 30 in. high and having a cross section of $\frac{1}{2}$ sq in.

16–4. What is the weight of the air in a room $20 \times 30 \times 12$ ft (temp 70°)?

16–5. The specific gravity of sea water is 1.03. How many cubic feet of water are displaced by a 400-ton ship?

16–6. Find the weight of 10 gal of gasoline.

16–7. Four cu ft of cork weigh 60 lb. What is the specific gravity of the cork?

16–8. What will be the weight of 40 ft of lead pipe, inside diameter 2 in., outside diameter $2\frac{5}{8}$ in.?

16–9. What weight of sheet copper $\frac{1}{16}$ in. thick will be required to line the sides and bottom of a rectangular tank 6 ft long, $2\frac{1}{2}$ ft wide, 3 ft deep, inside dimensions?

16–10. Calculate the weight of a cast-iron pipe 10 ft long, 4 in. bore, walls $\frac{1}{2}$ in. thick, having a circular flange at each end 9 in. diameter, $\frac{9}{16}$ in. thick.

16–11. One-half gallon of alcohol of specific gravity 0.79 is mixed with 1 gal of water of specific gravity 1.00. Compute the specific gravity of the mixture.

143. Pressure. In mechanics of fluids the term pressure is used in the sense of *intensity of pressure*, or *pressure per unit of area*. The unit of area, unless specifically stated otherwise, is understood to be a square inch. Thus the expression, "Air at 30 pounds pressure," signifies air which exerts 30 lb pressure on each square inch of surface, etc.

144. Pressure in liquids at rest. Suppose a small area, a, Fig. 235, to be horizontal and at a depth z below the surface of a liquid at rest. Assume all other sources of pressure removed. Then the area will be pressed vertically downward by a force equal to *the weight of the column of liquid above it*. This weight will be equal to the volume of liquid za times its density ρ, i.e., equals $za\rho$. If p be the pressure at any point of the area, then pa, or the total force on the area, must equal $za\rho$, i.e., p equals $z\rho$. Any change in the depth z, or in the density of the liquid, will produce a corresponding change in the weight of the column of liquid resting upon a given area, and hence upon the pressure. Thus we have the rule:

Pressure in a liquid varies with the depth and with the density of the liquid.

It is because of this greater pressure as the distance below the surface increases that dams, retaining walls of reservoirs, etc., are made thicker and stronger at the bottom, as in Fig. 248.

If the area a in the preceding illustration is not horizontal, the pressure at some parts is greater than at others, but the pressure at any given point in the area is, as before, dependent only upon the depth of this point below the surface and upon the density of the liquid.

The depth of liquid is often called *head*. In water, the density of a cubic inch is 0.0361 lb, and the pressure due to a head of 1 ft $= 1 \times 12 \times 0.0361 = 0.434$ lb per sq in.

The area of the surface on which the pressure acts does not affect the amount, which, as stated above depends only upon depth and density. This may be shown by a consideration of Fig. 236. The cubic foot of water in the cubical container weighs 62.5 lb; the pressure on the bottom equals 62.5 lb per sq ft, which divided by the 144 sq in. area of the bottom equals 0.434 lb per sq in. The cubic foot of water may be imagined to be 144 columns of water 1 in. square and 1 ft high. The weight of each column is

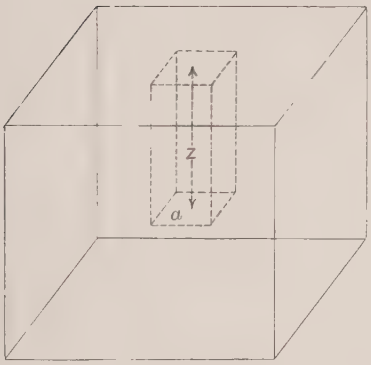


Fig. 235.

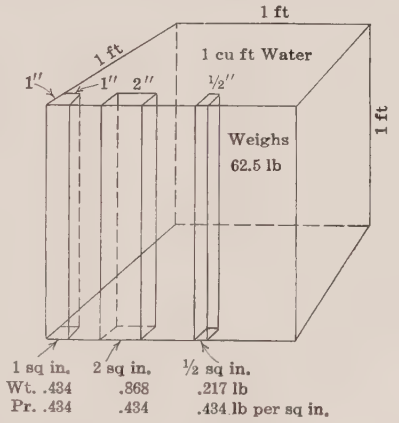


Fig. 236.

0.434 lb, and its pressure at the bottom equals 0.434 lb. The weight of a column with 2 sq in. cross section equals 2×0.434 equals 0.868 lb, which, distributed over 2 sq in. at the bottom equals the same pressure of 0.434 lb per sq in. Also the weight of a column having a cross section of 3 sq in. equals 3×0.434 lb, which, distributed over 3 sq in. equals the same unit pressure. Again, if the column has a cross section of $\frac{1}{2}$ sq in. its weight will equal $\frac{1}{2} \times 0.434$ lb, which, distributed over an area of $\frac{1}{2}$ sq in. equals 0.434 lb per sq in. **The unit pressure remains the same if the head is unchanged.**

145. Pressure equal in all directions. It is evident that the pressure at the depth z acts equally in every direction up, down, and at every angle. This must be true, for otherwise the liquid, being acted on by an unbalanced force in some direction, would move, which is contrary to the supposition that the liquid is at rest, i.e., in equilibrium.

146. Direction of pressure on the walls of the vessel. It may also be seen that the pressure of a liquid on the surface of the containing vessel *is always perpendicular to that surface*, for if it were not, there would be a component of the pressure tending to urge the liquid along the surface, and the liquid being free to move, would slide along the surface which, again, is contrary to the supposition that the liquid is in equilibrium.

147. Transmission of applied pressure through fluids. Pascal's Principle. If we take a tank or vessel full of water, as Fig. 237, having various apertures of the same size, fitted with water-tight pistons, and if one of these pistons be pressed inward with a force P, an *additional force* equal to P will be required at each of the other pistons to preserve equilibrium. Thus, if a force of 5 lb is applied to P, it will be necessary to apply a force of 5 lb at each

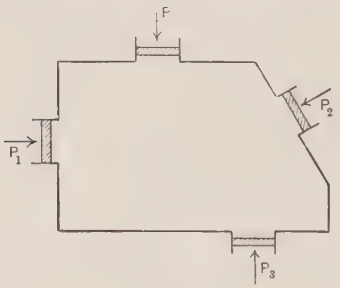


FIG. 237.

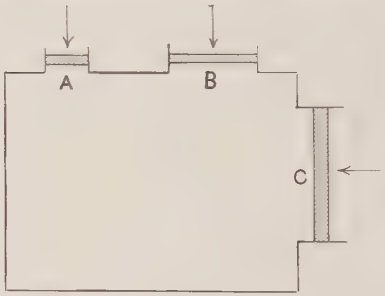


FIG. 238.

of the other pistons to prevent their moving, and this will be so no matter at what part of the vessel these pistons are placed. Thus, we see that any increase of pressure applied at one part is *transmitted equally* through the fluid. Furthermore if, as in Fig. 238, the pistons are of different areas, that at B being twice as large in area as that at A, and that at C being three times as large, then for every pound applied at A two pounds must be added for equilibrium at B, and three pounds at C in order to keep the unit pressure the same. In general, the forces at the different pistons will vary directly as the areas of those pistons, or if the pistons are of a circular cross section, the forces will vary as the squares of the diameters. Hence we arrive at the important principle.

Pascal's Principle. When pressure is communicated to any part of a fluid, it is transmitted equally on equal areas in all directions through the fluid.

In the illustrations above a liquid was referred to, but the student should notice that Pascal's Principle applies to *all fluids*, both gases and liquids.

148. Applications of Pascal's Principle. The principle just given explains the action of many pieces of apparatus and machines.

Consider two communicating vessels containing water, Fig. 239, one of which is much larger than the other. The vessels are fitted

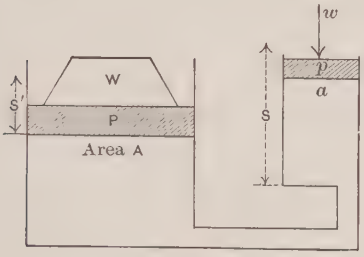


FIG. 239.

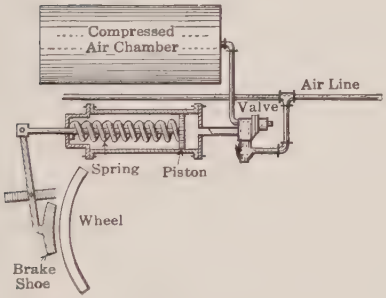


FIG. 240. Section of an air-brake. The triple valve admits air to the air chamber, allows air from the chamber to act on the piston head, or releases air from the brake cylinder to the atmosphere.

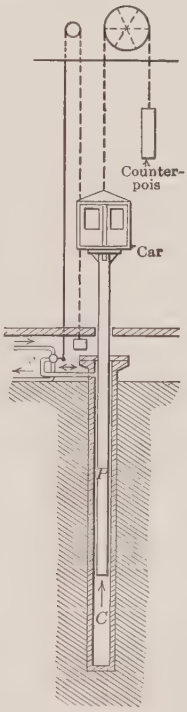


FIG. 241. Hydraulic elevator.

with pistons P and p, the areas of which are A and a. If now, weights W and w be placed on these two pistons respectively, so as to counterbalance each other, it will be found that

$$W : w = A : a$$

which is in accordance with Pascal's Principle.

We see, also, that if the piston p be pressed down through the

distance s, the water contained in the smaller vessel will be caused to pass into the larger, thus forcing up the piston P through some distance s', such that

$$a \times s = A \times s'$$

since the volume of water that is removed from one vessel is the same as that which enters the other vessel. Hence $a/A = s'/s$, that is, the simultaneous distances moved by the pistons vary inversely as their areas. Also

$$W \times s' = w \times s$$

that is, the *work done by* w = *work done by* W. This is the **Hydraulic Press.**

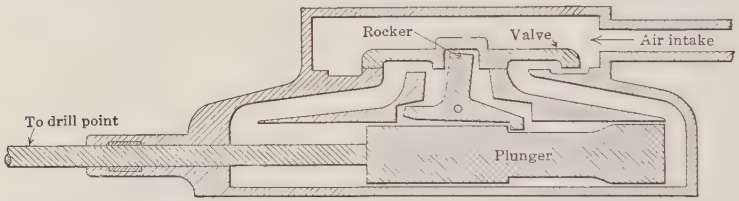


FIG. 242. Section of head of compressed-air drill. At the end of the stroke, the valve shifts and admits air at the opposite end to drive the plunger back. The expanded air escapes from the space around the rocker.

In order to make the action of the hydraulic press continuous and to provide the motion at W necessary in practice, the smaller piston is operated as the plunger of a pump by means of which water is pumped into the reservoir. Other forms of hydraulic and pneumatic machinery, as hydraulic jacks and elevators, pneumatic drills, riveters, airbrakes, etc., act upon a similar principle of transmission of pressure. The pumping up of bicycle and automobile tires with a small hand pump, and the rise of water to the same level in a closed system of pipes, may also be mentioned as instances in which Pascal's Principle applies. Pascal's Principle also applies to the U-tube described in Art. 142.

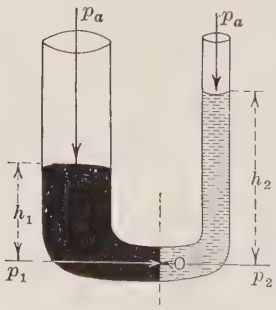


FIG. 243.

Figure 243 shows two open tubes of *different size* communicating by a flexible diaphragm O, one containing mercury, the other water.

The pressure at O is equal in all directions as the liquids are in equilibrium, therefore, $p_1 = p_2$. p_1 = atmospheric pressure, $p_a + h_1 \times \rho_1$. $p_2 = p_a + h_2 \times \rho_2$ where ρ_1 and ρ_2 = density of mercury and water respectively. Therefore, $p_a + h_1\rho_1 = p_a + h_2\rho_2$. Whence, $h_1\rho_1 = h_2\rho_2$ and

$$\frac{h_1}{h_2} = \frac{\rho_2}{\rho_1}$$

or the heights at which the two liquids stand in the tubes are inversely proportional to their densities.

PROBLEMS

16–12. Two communicating vessels contain fluid and are fitted with pistons, the diameters of which are 2 and 8 in. respectively. If a weight of 3 lb is placed on the smaller piston, what weight must be placed on the larger to preserve equilibrium?

16–13. A narrow vertical pipe is attached to a vessel which is fitted with a piston the area of which is 20 sq in. If the vessel and pipe contain water, find the height of the water in the pipe when a weight of 40 lb is placed on the piston.

16–14. Find the pressure in pounds per square inch on the bottom of a tank 72 in. deep and 30 in. in diameter. On the side adjoining the bottom. On the side 36 in. below the top.

16–15. In a hydraulic press, the diameter of the piston in the small cylinder is $1\frac{1}{2}$ in., the diameter of the piston of the large cylinder 16 in. If water is forced in under a pressure sufficient to cause a force of 80 lb for the entire area of the small piston, what will be: (*a*) The pressure per square inch on the larger? (*b*) The force on the larger?

16–16. A bicycle tire is cylindrical, $1\frac{1}{2}$ in. inside diameter, and 5 ft long. The opening for the pump is $\frac{1}{4}$ in. diameter. If air is forced in here with a force of one pound *for this area*, what will be: (*a*) The total force produced on the inside of the tire? (*b*) The pressure per square inch? (*c*) The force tending to burst the tire if atmospheric pressure is 15 lb per sq in.?

16–17. What is the pressure per square inch due to water alone at the bottom of a pond 30 ft deep?

16–18. The pipe *A*, Fig. 244, is 1 in. inside diameter. It is filled with water to a height $H = 4$ ft. What *weight* of mercury must be placed in *B* in order that, when valve *V* is opened, the liquids will remain at the same level?

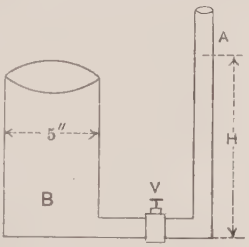


FIG. 244.

16–19. What will be the pressure at the bottom of a container 14 in. deep, filled half with water and half with gasoline?

16–20. Find (*a*) the weight of water in a tube 1-in. square and 2.3 ft high; (*b*) the pressure at the bottom of the tube; (*c*) the weight of water in a tank 20 ft square and 2.3 ft deep; (*d*) the pressure per square inch at the bottom of the tank.

149. Total pressure of a liquid on the base and sides of the containing vessel. The laws of fluid pressure stated in the preceding articles enable us to determine the total pressure on the walls of vessels, reservoirs, etc., whatever their shape.

(*a*) **On the base.** Three cases may be distinguished, according as the side walls are vertical or slope from the base outward or inward, as in Fig. 245.

Suppose the base in each case to be horizontal. If the sides are vertical, the total pressure is evidently equal to the weight of the liquid.

Next, suppose the sides to slope outward. Then the total pressure on the base is *less* than the weight of the liquid. It is equal to the weight of the liquid within the dotted lines. Explain what supports the liquid between the dotted lines and the sides.

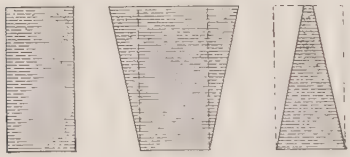


Fig. 245.

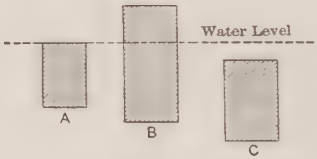


Fig. 246.

In the third case, where the sides slope inward from the base, the total pressure on the base is *more* than the weight of the liquid in the vessel. Explain how this can be so.

The general rule for the total pressure on a horizontal surface under a liquid, like the base of the vessel in either of the cases just given, is as follows:

The total pressure or force on a horizontal surface below a liquid is equal to the area times the depth times the weight of unit volume of the liquid.

(*b*) **On vertical walls.** If a surface is vertically placed in a liquid, we may find the total pressure on it as follows: Find the *average pressure* on the surface and multiply this by the total area. Pressure increases with depth, varying from zero at the top of a surface placed as A, in Fig. 246, to a maximum value equal to depth multiplied by density at the bottom. The average pressure for a

rectangular surface is, therefore, the pressure at a point half way from the upper edge of the rectangle to the lower edge. The student may work out for himself the average pressure and area for the other cases illustrated in Fig. 246. Where the surfaces are not rectangles, the varying pressure is not distributed according to depth alone, and the average pressure is evidently the pressure at the *center of gravity* of the surface. It is evident from inspection that this statement applies equally to rectangles, hence for surfaces of any form we may apply the following general rule:

The total pressure or force on any plane surface below a liquid is equal to the area in the liquid times the vertical distance from the surface of the liquid to the center of gravity of the area times the weight of unit volume of the liquid.

(*c*) **On surfaces inclined at an angle with the vertical.** Liquid pressure is *always perpendicular to the retaining surface* (Art. 146), hence the general rule stated under (*b*) may be applied to these surfaces also.

150. Center of pressure. When a surface is immersed in a liquid, every point on the surface is subjected to a pressure at right angles to the surface. All these pressures constitute a system of parallel forces, as Fig. 247 whose resultant may be found. In our study of liquid pressure we have learned how to find the *amount* of this resultant in certain cases, but we have not considered the point on the surface at which it acts. This point is called the **center of pressure,** and is defined as the point of action of the single force equivalent to the whole pressure exerted by a fluid on any plane surface with which it is in contact.

If the plane surface is *horizontal*, the center of pressure is the same as the center of gravity of the surface. Thus, the center of pressure on a horizontal rectangle, circle, triangle, etc., is at the center of gravity of the surface and a single force (equal to the total pressure on the area) applied at that point is equivalent to the distributed liquid pressure.

If, however, the surface is *vertical*, since the pressures increase proportionally to the depth and the total force on the surface depends on the shape and position of the surface, the resultant can only be obtained by calculation, but the *center of pressure is always below the center of gravity.* For if the pressures on equal areas on different parts of the surface were equal, the point of application of their resultant, the center of pressure, would coin-

cide with the center of gravity. But the pressure increases with the depth and therefore the center of pressure is necessarily below the center of gravity. The exact location of this point is found to be as follows:

1. With a rectangular surface whose upper edge is level with the liquid, the center of pressure is *two-thirds* of the way from the top on the line joining the middle points of the horizontal sides.

2. With a triangular surface whose *base is horizontal, and at the level of the liquid*, the center of pressure is at the *middle* of the line which joins the vertex with the middle of the base.

3. With a triangular surface whose *vertex is level with the water*, the center of pressure is in the line joining the vertex and the middle of the base, and *three-fourths* of the way from the vertex.

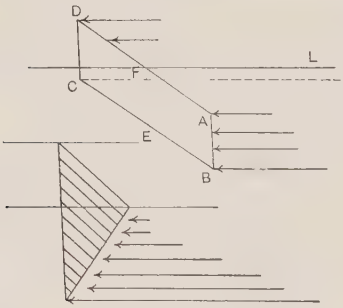


Fig. 247.

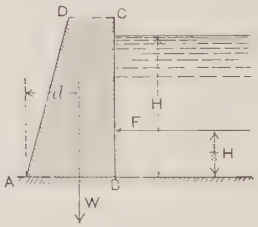


Fig. 248.

151. Stability of a retaining wall. Suppose $ABCD$, Fig. 248, represents a section of a retaining wall. The resultant water pressure F will act at a distance equal to one-third the depth of the water H from the bottom. Its tendency to overthrow the wall by tipping it over the edge at A will be measured by the moment $F \times H/3$. This moment will be opposed by the moment of the weight W of the wall, or $W \times d$. Evidently, in order that the wall may stand, $W \times d$ must be greater than $F \times H/3$.

PROBLEMS

16–21. Find the force due to the water on a rectangular surface 6 ft by 4 ft immersed vertically in water with the shorter side parallel to and 2 ft below the surface.

16–22. Find the weight of water and the pressure in pounds per square inch on the bottom of a tank 3 ft × 4 ft × 5 ft deep, if the tank is full. If one-half full.

16–23. A metal can, 8 in. in diameter and 12 in. deep, has a tube 1 in. in diameter and 4 ft high attached to the top. Find (*a*) the weight of water to just fill the can; (*b*) the weight to just fill the tube; (*c*) the pressure on bottom of can when the can only is full of water; (*d*) when both can and tube are full; (*e*) pressure on top of can when can only is full; (*f*) when both can and tube are full.

16–24. A flood gate is 6 ft wide and 10 ft deep. What is the force of the water on the gate when the water is level with the top?

16–25. What is the pressure in pounds per square inch at a depth of half a mile in sea water? Specific gravity 1.03.

16–26. The area of the small cylinder of a hydrostatic press is 3.5 sq in., and a force of 25 lb on the small piston will support 200 lb on the large piston. Find the area of the large piston. What is the unit pressure in the liquid?

16–27. A hole 6 in. square is made in a ship's bottom 20 ft below the water line. What force is required to hold a piece of wood tightly over the hole?

16–28. A mill dam is 40 ft long and the water is 15 ft deep, and the pond is 500 ft long. Find the force urging the dam down stream.

16–29. A house is supplied with water from a reservoir 240 ft above the ground. Find the pressure per square inch at a tap 25 ft above the ground.

16–30. A canal lock is 60 ft long, 20 ft wide, 10 ft deep. Find:

(*a*) The pressure per square foot on the bottom.
(*b*) The total force on a side.
(*c*) On the gate.
(*d*) Pressure per square foot on the gate half-way down; 8 ft down.

16–31. *ABC*, Fig. 249, stands in water with *BD* vertical. The apex *B* is 2 ft beneath the surface. $AD = DC = 4$ ft. $BD = 10$ ft. Compute the force against the surface.

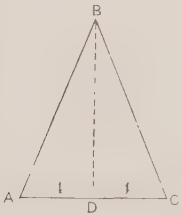


Fig. 249.

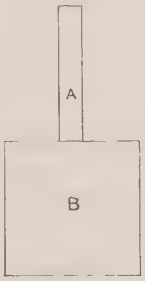


Fig. 250.

16–32. Find the pressure in a pipe line at a point 35 ft below the surface of the supply tank.

16–33. Find the height of the reservoir level above a faucet if the pressure at the faucet (assuming no flow) is 65 lb.

16–34. *A* and *B* are two cylinders placed as shown in Fig. 250. *A* is 2

ft high and 4 in. diameter. B is 2 ft high and 2 ft diameter. When both are filled with water compute:

(*a*) The force on the bottom of B.

(*b*) The force on a scale-pan supporting them if the cylinders alone weigh 25 lb.

16–35. A gauge on a pipe line is 5 ft above the floor, while another gauge on the same line is 36 ft above the same floor. If the first gauge reads 70 lb, what should the second one read?

152. Buoyancy of bodies immersed in a fluid. Archimedes' Principle. If we take a piece of iron weighing 1 lb and hang it wholly immersed in water from one scale of a balance, we find that it now seems to weigh *less than* 1 *lb*. The difference in the weight in the two cases is due to the resultant upward pressure of the water on the iron.

In Fig. 251 we have a rectangular block immersed in water with one surface horizontal. It is evident that the pressure on one vertical side of the block is just equal to that on the other side, and that there will be in general no resultant pressure tending to cause horizontal motion of the block. The resultant upward pressure is easily found as follows: The pressure on the top of the block is evidently that of a column of the liquid having AB for a base and AE for height, whereas the pressure on the bottom of the block is that of a column of the liquid having DC for a base and DE for height. The resultant upward pressure must be equal to the difference between the weights of these two columns, i.e., to the weight of a column of liquid $ABCD$, that is, *to the weight of the liquid displaced.* If the body be of irregular shape, a more general proof must be employed, but the result is the same. Hence,

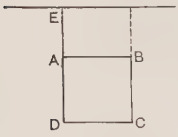


FIG. 251.

Archimedes' Principle. — A body immersed in a fluid loses a weight equal to the weight of the fluid displaced.

It is evident that Archimedes' Principle furnishes a convenient method of determining the specific gravity of a body and also the volume of irregular bodies. Thus, suppose a body weighing 80 lb in air, weighs only 60 lb when immersed in water. The loss of weight is 20 lb and, according to Archimedes' Principle, this is the weight of a volume of water equal to the volume of the body.

$$\text{Its specific gravity} = \frac{\text{weight of body}}{\text{weight of equal vol. water}} = \frac{80}{20} = 4$$

And since 1 cu ft of water weighs 62.5 lb, if a volume of water equal to that of the body weighs 20 lb the volume of the body is $20/62.5 = 0.32$ cu ft.

Archimedes' Principle applies to bodies immersed in a gas as well as to those in a liquid. Hence the *real weight* of all bodies (that is, their weight in a vacuum) is greater than their weight in air (which may be called their apparent weight), by the weight of the air displaced.

The resultant upward force upon bodies in a fluid is the source of the buoyant force on ships in the water, on balloons in the air, lighter-than-air dirigibles, etc. The amount of excess of this force over the weight of the floating body determines the added load these bodies can lift.

153. Hydrometers. Floating bodies sink in the liquid until they displace their own weight. The resultant upward force then

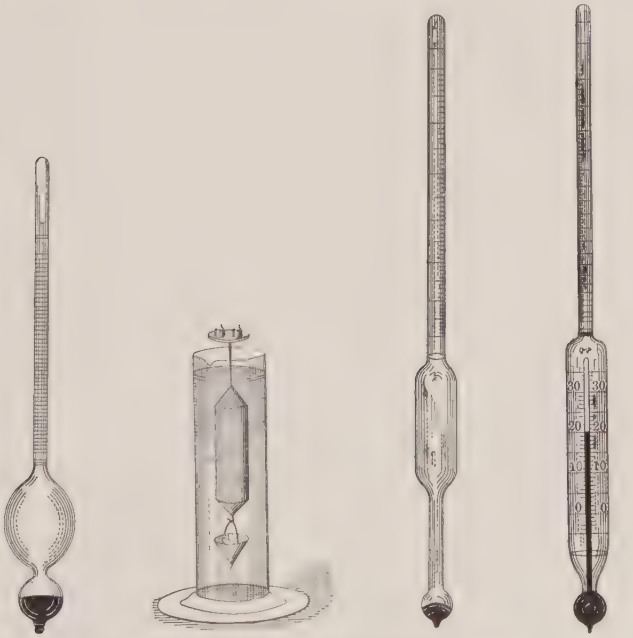


FIG. 252. Common types of hydrometers.

equals the downward force of gravity and the bodies are in equilibrium. It is evident that a floating body must sink deeper in a light liquid than in a denser in order to displace its own weight;

or, in other words, the volume of two liquids displaced must be inversely as the densities. This principle is the basis of the usual forms of **hydrometers,** or instruments which indicate the density of liquids by the scale readings on their stems. Common types of hydrometers are shown in Fig. 252. These instruments are calibrated by placing them in liquids of known density and marking upon the stem the point to which they sink, that for pure water being marked 1.000. To provide greater sensitiveness, a single instrument is designed to measure a small range of densities only. Special instruments are thus made for testing alcohol, milk, the acid in storage batteries, the anti-freeze solution in radiators, etc.

PROBLEMS

16–36. A barge with vertical sides, floating in fresh water, is 30 ft long and 20 ft wide. An elephant is driven upon the barge and the barge is then found to have sunk 2 in. Find the weight of the elephant.

16–37. A man weighs 150 lb and his volume, exclusive of his head, is 4000 cu in. How many cubic inches of cork (specific gravity 0.25) are required to keep the man floating with his head above water?

16–38. What fractional part of an iceberg (specific gravity 0.918) will float above sea water (specific gravity 1.03)?

16–39. A piece of metal weighs 36 lb in air and 32 lb in fresh water. What will it weigh in sea water of density 1.03?

16–40. A cylindrical post 30 ft long and 8 in. in diameter, has a weight of 880 lb. It has 6 ft of its length projecting vertically out of water. What force is required, and in what direction, to keep it in this position?

16–41. An iron ball weighing 125 lb is suspended under water. What is the tension in the suspending string?

16–42. A piece of brass weighs 21 oz in air, and 18 in water. Find its volume, density, and specific gravity.

16–43. A cube of metal, $2\frac{1}{2}$ in. on each edge, weighs $4\frac{1}{2}$ lb; find weight of the cube in water and its specific gravity.

16–44. How far must a float $14 \times 10 \times 6$ in. be submerged to secure a buoyancy of 10 lb in order to operate a shut-off mechanism? Weight of float, 2.5 lb.

154. Head. In Art. 144 it was shown that the pressure in a liquid depends upon the depth. In hydraulics the *vertical depth* of any point below the surface plane of the water is called the **head.** If h represents the head, p the pressure (i.e., unit-pressure), and w the weight of a cubic unit of water,

$$p = wh, \quad \text{and} \quad h = \frac{p}{w} \tag{80a}$$

Any unit-pressure p can be regarded as produced by a head h, which is commonly called the **pressure head.** In engineering work p is usually expressed in pounds per square inch, h in feet, and 62.5 lb is used as the weight of 1 cu ft of water. Therefore,

$$p = \frac{62.5}{144} h = 0.434\, h, \quad \text{and} \quad h = 2.304\, p \tag{80b}$$

1 foot head corresponds to 0.434 pound per square inch.
1 pound per square inch corresponds to 2.304 feet head.

155. Velocity of flow from an orifice. An expression for the theoretic velocity of flow from an orifice in the bottom or side of a vessel containing water was established in 1644 by Torricelli.

The theoretic velocity of flow from an orifice is the same as that acquired by a body after having fallen from rest in a vacuum through a height equal to the head of water on the orifice.

The velocity v acquired by a body falling freely from rest through the distance h is given by the expression $v = \sqrt{2\,gh}$. (Art. 62.) Therefore by Torricelli's theorem, representing the velocity of the issuing water by v, and the head by h,

$$v = \sqrt{2\,gh}, \quad \text{and} \quad h = \frac{v^2}{2\,g} \tag{81}$$

The first equation gives the theoretic velocity of flow from the orifice, due to a given head h, the second the theoretic head that will produce a given velocity v. The expression $v^2/2\,g$ is commonly termed the **velocity head.** In any actual case of flow from an orifice, the velocity head $v^2/2\,g$ is always less than the pressure head h because of loss of head in friction and impact.

If there is pressure other than atmospheric on the surface of the water in the vessel, the velocity of discharge into the atmosphere from the orifice is increased. This increase is best taken into account by reducing the given pressure to its equivalent head and adding this to the head of actual water on the orifice. Thus suppose the surface of the water is subjected to a pressure p' from a piston or other device, then the equivalent head h' is, $h' = 2.304\, p'$, and if there is also a head of actual water h, the theoretic velocity from the two causes is expressed by the equation,

$$v = \sqrt{2\,g\,(h + h')} \tag{81a}$$

The velocity of flow is also increased when the discharge is into a pressure less than atmospheric. This is best taken into account by determining the head h'' produced by the reduction in pressure. Suppose the pressure into which the water discharges is p''. Then $h'' = 2.304\ (14.7 - p'')$, and if h is the head of actual water, the theoretic velocity v is now expressed by the equation,

$$v = \sqrt{2\,g\,(h + h'')} \tag{81b}$$

156. Quantity discharged. When water is flowing from an orifice in the side of a vessel, the particles of water outside a region relatively near the orifice will be moving comparatively slowly. As the particles approach the orifice their velocity increases. Particles directly opposite the orifice will move straight for the orifice and go out; those near the wall in which the orifice is placed will move with increasing velocity toward the orifice and must change their direction of motion nearly 90°; particles situated between these extremes will move in curved paths toward the orifice. Because of their inertia, the particles will not turn the corner at the edge of the orifice sharply, and as a general effect of all the movements of its particles the jet is caused to contract in area of cross section after the orifice is passed. The ratio of the area of the jet at its smallest section to the area of the orifice is known as the **coefficient of contraction.** It is the number by which the area of the orifice is to be multiplied to give the area of the jet at its smallest section. An average value for the coefficient of contraction is 0.62.

It is found also that the velocity at the least cross section of the jet is less than the theoretic velocity of flow from the orifice. The ratio of the actual velocity at the least cross section to the theoretic velocity of flow from the orifice is called the **coefficient of velocity.** It is the number by which the theoretic velocity must be multiplied to give the actual velocity at the least cross section of the jet. An average value for the coefficient of velocity is 0.98.

From the preceding it will be seen that the actual quantity of water discharged through an orifice is less than the quantity that might be expected from theoretical considerations, both because of the contraction of the jet and the fact that the actual velocity is less than the theoretic velocity. The product of the coefficient of contraction and the coefficient of velocity, called the **coefficient of discharge,** is the number by which the theoretic discharge must

be multiplied to obtain the actual discharge. An average value of the coefficient of discharge is therefore 0.62×0.98 or 0.61.

Let Q represent the quantity or rate of discharge in cubic feet per second (cfs), A the area of the orifice in square feet, v the velocity in feet per second, h the head in feet, then the theoretical velocity will be:

$$v = \sqrt{2gh} \tag{81}$$

and the *theoretical* discharge will be:

$$Q = Av \tag{82}$$

Observe that the discharge stream can be visualized as a cylinder, with a circular area = A, and a length = the distance a particle would travel in one second = v; the volume of the cylinder would be Av. If the discharge were from a pipe its volume would equal the volume of the inside of the pipe as far back as the water would travel in a second.

The *actual* discharge then is

$$Q = 0.61\, A \sqrt{2gh} \tag{83}$$

Example.

Find the probable discharge in gallons per minute (gpm) from a 1-in. diameter orifice under a constant head of 9 ft.

Solution.

$$A = 1^2 \times \frac{0.7854}{144} = 0.005 \text{ sq ft}$$

$$Q = 0.61\, A \sqrt{2gh} = 0.61 \times 0.005 \times \sqrt{2 \times 32 \times 9}$$

$$Q = 0.073 \text{ cfs}$$

Since there are 7.48 gal in a cu ft (see table in Appendix)

$$Q = 0.073 \times 7.48 \times 60 = 32.8 \text{ gpm} \quad \textit{Ans.}$$

157. Flow in pipes. When a pipe is filled with water at rest, the pressure at any point depends only on the head above that point. When, however, the water is in motion through the pipe the pressure becomes less than that due to the head. The simplest way of measuring the pressure at any point is by noting the height to which the water will rise in a vertical tube inserted at the point. Such a tube is called a **piezometer.**

Thus in Fig. 253, if the valve in the pipe is closed, the sections 1 and 2 are both filled with water *at rest*, and the water will rise in both vertical tubes to the same level as that of the water surface in the reservoir R. The pressures in the sections will be those due to the hydrostatic heads H_1 and H_2. But if the valve is opened, the water levels in the vertical tubes will fall, and the mean pres-

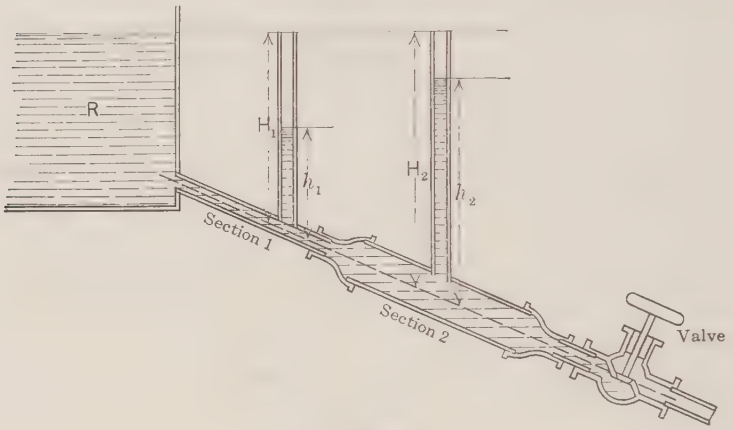


FIG. 253.

sures in the two sections will be those due to the pressure heads h_1 and h_2. If the mean velocity of flow in section 1 is v_1 and in section 2 is v_2, the velocity heads are $v_1^2 \div 2g$ and $v_2^2 \div 2g$, respectively, and by the law established in 1738 by Bernouilli that, "**at any section of a tube or pipe, under steady flow without friction, the pressure-head plus the velocity-head is equal to the hydrostatic-head,**" assuming the sections of pipe to be smooth, we may write,

$$H_1 = h_1 + \frac{v_1^2}{2g}, \quad \text{and} \quad H_2 = h_2 + \frac{v_2^2}{2g} \tag{84}$$

From the preceding equations it is evident that the pressure head at any section decreases when the velocity of the water increases.

The **Pitot tube** is a device for measuring the velocity in a stream by means of the velocity head that it will produce. In its simplest form the Pitot tube consists of a tube open at both ends and so bent that the short arm is at right angles to the larger arm. The mouth of the short arm is so placed as to face the current directly.

The water then rises in the vertical arm to a height h above the surface of the flowing stream and this height is equal to the velocity head $v^2/2g$. The actual velocity v is therefore approximately equal to $\sqrt{2gh}$.

If a second tube is arranged so as to face sideways to the current, the height in this tube records only the static pressure head, without the effect of the velocity head.

158. Friction in pipes. The loss of head in any pipe in actual use is represented by the total head minus the velocity head $v^2/2g$ at the orifice for discharge. In a straight pipe of uniform size this loss of head is made up of the loss of head at entrance plus the loss of head in friction. The loss of head at entrance is usually relatively small, whereas the loss of head in friction may be very large if the pipe is long. From tests with pipes of different sizes and lengths with different velocities of flow there have been deduced the following **approximate laws for the loss of head in friction:**

1. The loss of head in friction is directly proportional to the *length* of the pipe.
2. It is inversely proportional to the *diameter* of the pipe.
3. It varies approximately as the square of the *velocity* of flow.
4. It is independent of the *pressure* of the water.
5. It increases with the *roughness* of the interior surface of the pipe.

These laws should be compared with the laws for fluid friction in Art. 99.

When there is a sudden enlargement of section in a pipe, there is loss of head from impact. Quick bends in a pipe produce loss of head for the same reason. When there is a sudden contraction of section in the direction of the flow, the water suffers a contraction similar to that in the discharge from an orifice, and hence in the expansion to fill the section following, a loss of head results.

The diagram in Fig. 254 suggests a laboratory arrangement in which water flows from a tank, through a pipe, with a glass piezometer tube at points 1 and 2. The rate of discharge is controlled by a valve at the end of the pipe. The level in the tank is kept constant by a ball float.

The following terms may be used to define the conditions at any point along the line. The vertical distance of the reservoir surface above some datum elevation, as a table top, is the *total*

hydraulic head H. At each location the height of the pipe above the datum plane is the *elevation head* h_e. The height of water in the tube (= the static head at the point) is the *pressure head*

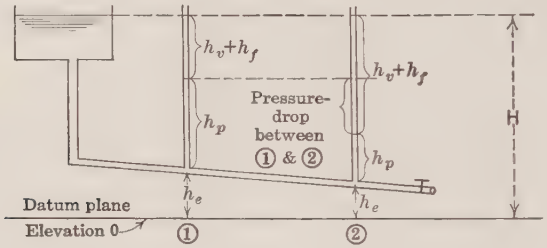


FIG. 254.

h_p (= 2.304 p). The drop in head corresponding to the velocity of the flowing water is the *velocity head* h_v ($=v^2/2\,g$). The drop in head required to overcome the friction is the *friction head* h_f. The total head H is constant.

$$H = h_e + h_p + h_v + h_f \tag{85}$$

A change in any of the subheads requires a change in the opposite direction in some other subhead in order to keep the sum constant

The pressure loss or "drop," due to flow, between the reservoir and either station is the sum of the velocity head and the friction head to that point, leaving available the pressure head. The velocity is the same at both points in this case, since the pipe is the same size throughout. The pressure drop between the two points is the difference between the friction heads at each point.

The pressure head at station 2 is less than at station 1 because of the drop; in this case it is not so much less as it would be if the pipe were level, because of the slope, i.e., the elevation head is less.

The friction loss in pipes, due to the pressure used in overcoming the frictional resistance to flow, varies with several conditions, as stated before. Pressure loss has been determined experimentally and tabulated for different kinds and sizes of pipe for different rates of discharge. The pressure loss in the tables includes both the friction and the velocity head, and is expressed sometimes as pressure in pounds per square inch, and sometimes as equivalent feet head (h = 2.304 p). The following tables, compiled from experimental data from various sources, are typical of more complete data in reference books on hydraulics.

FLOW OF WATER IN BRASS PIPES

Pressure Loss (Pounds per sq in. per 100 ft length)

Discharge Gal per min	Diameter of pipe in inches						
	¾	1	1¼	1½	2	2½	3
5	3.0	0.70	0.20	0.10	0.03	0.02	0.0
10	8.9	2.73	0.72	0.35	0.11	0.05	0.0
20	31.6	9.79	2.50	1.20	0.38	0.13	0.05
30	70.0	20.8	5.45	2.56	0.90	0.27	0.12
40	....	35.5	9.25	4.33	1.36	0.43	0.20
50	....	55.0	14.10	6.59	2.05	0.71	0.30
60	....	80.0	19.5	9.18	2.87	0.99	0.41
70	....	....	26.4	12.3	3.84	1.32	0.55
80	....	....	33.8	15.6	4.92	1.89	0.70
90	....	....	41.6	19.5	6.11	2.10	0.88
100	....	....	51.0	23.8	7.43	2.55	1.07

FLOW OF WATER IN IRON PIPES

Pressure Loss (Pounds per sq in. per 100 ft length)

Discharge Gal per min	Diameter of pipe in inches						
	¾	1	1¼	1½	2	2½	3
5	3.8	0.80	0.25	0.12	0.035	0.026	0.0
10	13.0	3.16	1.00	0.47	0.120	0.050	0.0
20	50.4	12.30	3.75	1.66	0.420	0.190	0.06
30	110.0	27.50	7.70	3.75	0.910	0.40	0.13
40	....	48.00	14.20	6.52	1.60	0.68	0.23
50	....	75.0	22.00	10.00	2.44	1.03	0.35
60	....	110.0	31.3	14.0	3.50	1.48	0.50
70	....	....	42.3	20.0	4.80	1.91	0.68
80	....	....	55.0	25.0	6.30	2.30	0.88
90	....	....	69.4	31.5	7.85	2.81	1.06
100	....	....	85.0	39.0	9.46	3.70	1.31

Pressure loss in pipe fittings is usually expressed in terms of the equivalent length of straight pipe of the same size and material. For a pipe of 1-in. diameter the loss in a right elbow is equivalent to about 1½ ft of straight pipe. The friction in a 45 deg elbow is

about one-half as much as a right elbow, that in a tee, running "through," is also about one-half, and in a tee using the "side outlet" it is twice as much as the right elbow.

Pipe size is given by naming the approximate *inside* diameter. Standard diameters actually are a slight amount larger than the nominal size; for example, one-inch pipe has an actual diameter of 1.049 in. Fittings have the same *nominal* size *as the pipe they fit.* Thus, a 1-in. pipe (made of metal about $\frac{1}{8}$ in. thick) measures *about* 1 in. on its inside diameter, about $1\frac{1}{4}$ in. outside diameter; and a 1-in. fitting will have a $1\frac{1}{4}$-in. diameter inside and a $1\frac{1}{2}$-in. diameter outside.

Small water pipe is made of copper, brass, or steel. The use of wrought-iron pipe has been largely discontinued, although genuine wrought-iron pipe can be obtained. So-called black iron pipe is now usually made of mild steel. Cast iron is used for large sizes and for drain pipe.

Example 1.

In a laboratory arrangement as in Fig. 254, a point (1) is 5 ft above the floor and a second point (2) is $4\frac{1}{2}$ ft above the floor. The water stands at a height of 26 in. in piezometer (1) and 11 in. at (2). Find the pressure drop between the two points.

Solution. The velocity is not given, and may be neglected since it must be the same at both points. Assume h_f at (1) = 0, as no data for that are given, and the problem deals only with conditions "down stream" from point (1).

Then at (1) $$H = h_e + h_p + h_f$$
$$= 60 + 26 + 0 = 86 \text{ in.}$$

And at (2) $$86 = 54 + 11 + h_f$$

From which $$h_f \text{ at (2)} = 86 - 65 = 21 \text{ in.}$$

The difference between the two points = $21 - 0 = 21$ in. *Ans.*

That is, the level of the water in the second tube is 21 in. lower than in the first tube.

Example 2.

Referring to the table of pressure losses, what will be the probable discharge in gallons per minute from a $1\frac{1}{4}$-in. iron pipe 200 ft long if the inlet is 196 ft above the outlet?

Solution. Pressure available to be used up in causing flow, due to 196 ft head = $p = 0.434\ h = 0.434 \times 196 = 85.2$ lb per sq in. The

table gives pressure loss per 100 ft length of pipe, and friction is proportional to the length. For 200 ft the pressure available is 85.2 lb. $85.2 \div 2 = 42.6$ lb for each 100 ft. In the column for $1\frac{1}{4}$-in. pipe, 42.3 is the loss for a discharge of 70 gpm. Therefore, probable discharge = 70 gpm. *Ans.*

Note. If the pipe were only 1-in. diameter, discharge would be between 30 and 40 gpm.

159. Conversion of the energy of water. Every pound of water at a height of h feet above the level at which its energy is to be used has a potential energy of h foot-pounds. If it falls freely through the height h, its potential energy is changed into kinetic energy, and at the lower level if the velocity is v feet per second, its kinetic energy is $v^2/2\,g$ foot-pounds. The potential energy lost is equal to the kinetic energy gained, or $h = v^2/2\,g$ (Art. 90). Useful work may be obtained from the h foot-pounds of potential energy available for each pound of water by permitting the water to descend through a pipe, thus producing pressure at the lower level which may be used to turn a wheel or drive a piston, or by utilizing some of the kinetic energy acquired in falling. In the latter case, the velocity of the moving water is decreased and kinetic energy given up by the moving water is transferred to a water-wheel or turbine. The work performed by hydraulic motors may range from 25 to over 90 per cent of the theoretic energy of the falling water.

The discharge of small streams and the quantity of water supplied to hydraulic motors are very commonly measured by allowing the water to flow through a rectangular notch called a **weir** and measuring the depth of water in the weir. Allowance must be made both for contraction and for friction, by determining first the coefficient of discharge for the weir under the given working conditions.

PROBLEMS

16–45. What pressure in pounds per square inch corresponds to a head of 180 ft? How many feet head correspond to a pressure of 90 lb per sq in.?

16–46. What is the theoretic velocity of a jet under a head of 40 ft?

16–47. What is the theoretic velocity of outflow from an orifice on which the head of water is 16 ft if a moving piston is applying at the same time a pressure of 1200 lb per sq ft on the surface of the water?

16–48. Water under a head of 200 ft flows into a boiler. If the boiler gauge reads 30 lb per sq in., what is the theoretic velocity of the inflowing water?

16–49. What is the actual discharge in gallons per second from a circular orifice 2 in. in diameter on which the head is 12 ft?

16–50. In the laboratory arrangement as in Fig. 254, a point A is 6 ft above the floor and B, 20 ft distant, is $4\frac{1}{2}$ ft above the floor. The water stands 26 in. at A and 11 in. at B. Find the pressure drop between A and B, and the rate of drop per 100 ft of pipe.

Note. The following problems refer to the tables in Art 158:

16–51. Find the pressure loss (*a*) in a 1-in. diameter iron pipe 100 ft long when discharging 10 gpm. Same when discharge is 20 gpm. To what head in feet does each of these answers correspond? (*b*) Same for brass pipe.

16–52. Find pressure loss in 100 ft of $\frac{3}{4}$-in. iron pipe to discharge 10 gpm. Same for 1-in. diameter. Same for 2-in. diameter.

16–53. How many gallons per minute can be discharged through a $1\frac{1}{4}$-in. iron pipe 100 ft long under an available pressure of 7.7 lb per sq in.?

16–54. How many gallons per minute can be discharged through a $1\frac{1}{4}$-in. iron pipe 200 ft long if the inlet is 30 ft above the outlet?

16–55. A 1-in. iron pipe is to discharge 20 gpm. How much pressure will be required per 100 ft? At what elevation must the inlet be above the outlet if the length is 100 ft?

16–56. What head will be required on a 100-ft length of $1\frac{1}{2}$-in. pipe to deliver 20 gpm?

16–57. A house requires a maximum of 10 gpm from a spring 600 ft distant and at an elevation of 45 ft above the outlet of the pipe. What size pipe can be used?

16–58. If a gauge reads 40 lb on a pipe line, what will be the pressure 100 ft away at a point 10 ft lower with the water flowing at 20 gpm? Diameter of pipe 1 in.

16–59. Water flows through 150 ft of horizontal pipe and up vertically through 50 ft more of pipe line. A gauge at the start of the 150 ft run reads 43 lb per sq in., while a gauge at the point 50 ft up reads 8 lb per sq in. Compute the friction loss in terms of lb per sq in. per 100 ft of pipe.

160. Gases. Gases, like liquids, have weight and such perfect freedom from molecular friction that they obey the laws of of Pascal and Archimedes.

Gases, however, differ from liquids in three important respects:

1. They behave as if their molecules repelled each other.
2. They are very compressible.
3. They expand much more rapidly when heated.

The proof of the first of these statements is seen in all the familiar

cases where a gas expands, as its pressure is reduced. A gas must be kept in a vessel enclosed on all sides or it will expand indefinitely.

The proof of (2) is also given by many common experiments, some of which will be considered presently.

The density of a gas is dependent upon both pressure and temperature. If no specifications are given, it usually may be assumed that *standard conditions* are intended, i.e., a pressure of 30 in. of mercury and a temperature of 32 F. The specific gravity of a gas is usually referred to that of dry air as a standard.

161. Atmospheric pressure. That the air has weight was not known until about 1620, when the work of Galileo, Torricelli, Pascal, Guericke, and others established the fact. The weight of the air may be demonstrated in the same way as the weight of anything else, that is, by weighing a vessel full of air and then

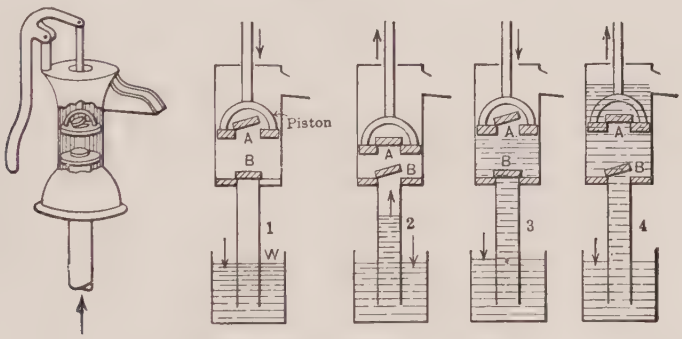


FIG. 255. Common type of pump.

the same vessel empty, and observing the difference. The weight of the air has the most important consequence upon the conditions not only of human life, but also upon the conduct of very many practical operations and scientific experiments. All the phenomena commonly spoken of under the term "suction," are due to inequalities of atmospheric pressure, i.e., the rise of water from the cistern into the barrel of a common pump. The operation of siphons, of various types of air-pumps, etc., depends upon atmospheric pressure.

162. The barometer. Torricelli (an Italian, died 1647) devised an experiment to show the weight of the air, which may be performed as follows: Fill a glass tube at least 31 or 32 in. in length, open at one end and closed at the other, carefully with mercury

and, placing the thumb over the open end, invert the tube with this end under the surface of some mercury contained in a shallow vessel. On removing the thumb, the mercury will be found to sink somewhat and to come to rest with its surface at a level about 30 in. higher than the level of the mercury in the vessel. It is evident that the pressure of the column of mercury in the tube is equal to the height of the column times the weight of a unit volume of the mercury, or about 15 lb per sq in. The atmospheric pressure on the surface of the mercury in the vessel therefore must also be about 15 lb per sq in. to maintain the mercury in the tube. And since the pressure at the mouth of the tube must always be the same in all directions, if atmospheric pressure increases, the mercury must rise in the tube until its weight becomes sufficient for equilibrium; and, if atmospheric pressure decreases, the column of mercury must sink for a similar reason. Torricelli's apparatus is, therefore, a simple **barometer,** or instrument for measuring atmospheric pressure.

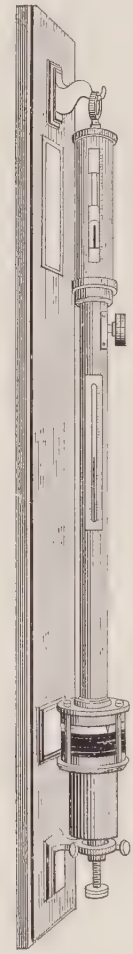
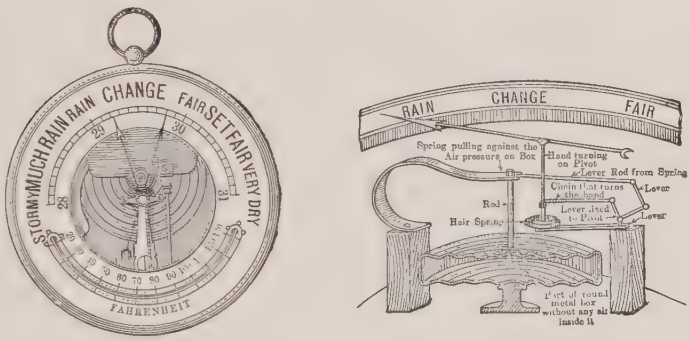


Fig. 256. Standard mercury barometer.

The common form of standard barometer is shown in Fig. 256. As the scale for reading the height of the column of mercury is rigidly attached to the case, the first operation in determining atmospheric pressure with this form is to set the surface of the mercury in the cistern at *zero* of the scale. This is done by means of a screw, which raises or lowers the flexible bottom of the cistern until the mercury is just in contact with the tip of the ivory point which marks the zero of the scale.

The so-called **Aneroid Barometer,** Fig. 257, consists of a thin metallic box tightly closed on all sides, and having a top of corrugated form which is easily compressed to a slight degree. The air in the box is slightly exhausted so that the variations of the pressure of the air outside cause the top of the box to move in or out through small distances. A suitable set of small gears and levers transmits these motions to a pointer moving over a scale. The scale is marked off in

the first place by comparing the aneroid with a standard mercury barometer, and the instrument requires to be frequently so compared to be reliable.



(*a*) (*b*)

FIG. 257. Aneroid barometer.

163. Standard pressure. Absolute pressure. As has been stated, the mercury will stand in a barometer tube at a height of about 30 in. or 76 centimeters. This is, however, subject to considerable variation due to the changes in weather, etc. **Standard pressure** means a pressure of 30 in. at a temperature of 32 F. This is the pressure to be assumed in the absence of any specifications, and is the pressure to which all scientific data are reduced in tables, etc.

Pressures expressed in heights of the column of mercury which atmospheric pressure will support may readily be reduced to pressures in pounds per square inch, by multiplying height by the density expressed in proper units. (See Art. 147.) Thus, 30 in. of mercury corresponds to 14.7 lb per sq in. (in round numbers, 15 lb per sq in.).

Ordinarily, pressures are stated in pounds per square inch, neglecting the pressure of the atmosphere. This is called ordinary pressure, or *gauge pressure*, since pressure gauges are graduated with the zero on the scale corresponding to ordinary atmospheric conditions. See Art. 166. For many computations in connection with pressure in closed systems, as steam boilers, refrigerators, etc., which are not open to the atmosphere, it is necessary to use **absolute pressure,** i.e., gauge pressure + atmospheric. **Absolute pressure = gauge pressure + 15 lb, or more precisely,** +

the barometer reading at the time. For some computations the barometer is assumed to be 30 in. = 14.7 lb per sq in. For many computations 15 lb is precise enough, and is considered sufficiently exact for use in problems in this book.

164. Boyle's Law. Dalton's Law. We have seen that the characteristic qualities of a gas are its expansibility and compressibility.

Experiment shows that if the volume of a gas be *doubled*, its *absolute pressure is reduced one-half*; if the volume be made *one-third* as great, the *pressure* will become *three times* as great, etc. In general, we have the statement known as

Boyle's Law. The absolute pressure of a gas varies inversely as its volume, when the temperature remains constant. This is sometimes called Marriotte's Law. Thus in symbols:

$$\frac{p_1}{p_2} = \frac{V_2}{V_1} \quad \text{or} \quad p_1 V_1 = p_2 V_2 \tag{86a}$$

The following two relations are also based on Boyle's Law. Since the *mass* of the gas remains the same, it follows that the density of a gas must increase as its volume decreases, and vice versa; that is, the density varies inversely as the volume. Therefore **the pressure of a gas varies directly as its density, temperature remaining constant.**

In symbols:
$$\frac{p_1}{p_2} = \frac{\rho_1}{\rho_2} \tag{86b}$$

Again, since weight depends upon density, **the pressure of a gas varies directly as its weight in a given volume, temperature remaining constant.**

In symbols:
$$\frac{p_1}{p_2} = \frac{W_1}{W_2} \tag{86c}$$

It should be added that these statements are for what is called a *perfect gas*. All gases are assumed to be perfect gases for purposes of computation, although none of them is so in fact, and none obeys the laws of gases absolutely. The variations are so small, however, as to be discernible only by the most delicate experiments, and are negligible in practical work.

Dalton's Law. This is an extension of Boyle's Law for a mixture of different gases. If several gases which do not act chemically on each other are placed in a vessel, the pressure on the sides of the vessel is the sum of the pressures due to the different gases. Thus, if the pressures of the different gases are p_1, p_2, p_3, p_4, etc., the intensity of the total pressure exerted by the mixture is $p_1 + p_2 + p_3 + p_4$, etc.

Dalton's Law may be stated as follows:

When a mixture of several gases at the same temperature is contained in a vessel, each gas produces the same pressure as if the others were not present.

In all the foregoing relations the pressure p is in absolute units. Other gas laws will be treated in Art. 170.

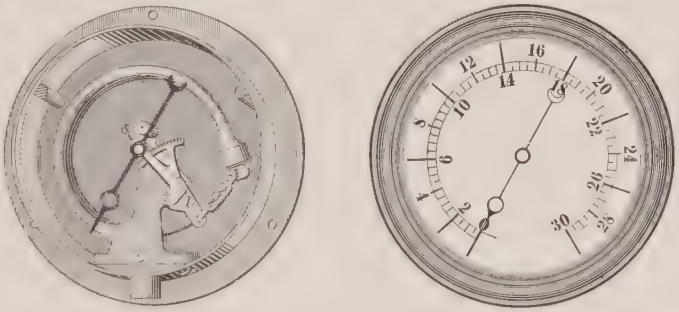


FIG. 258. Common type of pressure gauge.

165. Measurement of gas pressures. Pressure gauges and manometers are instruments for measuring the pressure of gases or vapors in closed systems. The ordinary commercial form of vacuum and pressure gauges is shown in Fig. 258. The pressure gauge is graduated in units of pounds per square inch. The vacuum gauge is usually graduated in inches of mercury. Thus a vacuum equal to 30 in. of mercury would correspond to −15 lb gauge pressure, and to 0 lb absolute pressure.

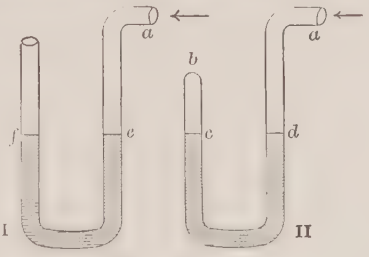


FIG. 259.

Two forms of simple manometers, the open and the closed tube types, are shown in Fig. 259.

In use, type I is filled with some liquid of known density, and the end a is connected with the closed vessel or pipe, the pressure within which is to be determined. The difference in level of the liquid in the two arms indicates the difference between the pressure exerted by the atmosphere and the pressure exerted by the confined fluid within the vessel. It may therefore be used for pressures either greater or less than atmospheric pressure. The actual pressure is obtained by computation from the barometer reading and the observed difference in reading of the manometer columns.

It should be noted that the difference between the levels in the manometer gives, when reduced to pressure units, the effective pressure within the vessel or "gauge pressure," and to this must be added the atmospheric pressure to obtain the *absolute* pressure. It is the former of these that is given by the ordinary steam gauge as used on boilers, etc.

Type II requires the use of *absolute* pressures, and is used for the measurement of higher pressures. The end b is closed and contains a column of air above the *mercury* in the tube. In using such a manometer, readings are first taken of the volume of air (usually expressed simply as lengths of tube occupied by air), and the difference of levels of the mercury when the end a is open to the air. Connection is then made at a with the closed vessel. The pressure of the fluid in this vessel will change the level of the mercury. Readings are then taken of the volume of the air and the difference of levels of the mercury in the tubes. From the two sets of readings the volume of the air and the pressure upon it (the reading of the barometer being known) in the beginning and the volume of the air under the unknown pressure of the vessel are known. The unknown pressure may then be computed by the application of Boyle's Law. This, corrected for the difference of level of the mercury, will give the pressure in the closed vessel.

Example.

In a closed arm manometer under atmospheric pressure, the mercury stands 6 in. higher in the open than in the closed arm, and the air column in the closed arm measures 8 in. When connected to a water tank the pressure on the open arm causes the mercury to drop in that arm 2 in. See Fig. 260(a) and (b). Find the water pressure.

Solution. The pressure on the confined air in the first case is 18

lb per sq in. absolute (i.e., 15 lb atmospheric + 3 lb due to 6 in. of mercury, at 0.49 or practically $\frac{1}{2}$ lb per in.). When the mercury drops 2 in. in the open arm it also rises 2 in. in the closed arm, so that the new mercury difference is 2 in., and the new air column is 6 in.

The volume of the air is now $\frac{6}{8}$ as much as before, and by Boyle's Law its pressure must be $\frac{8}{6}$ as much as before; $\frac{8}{6} \times 18 = 24$ lb abs., on the air column.

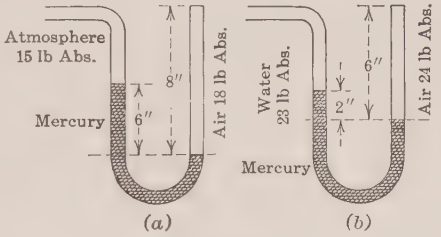


Fig. 260.

This includes 1 lb due to the 2-in. mercury column in the open arm.

Therefore the pressure on the open arm = 24 − 1 = 23 lb abs. The water pressure = 23 − 15 = 8 lb gauge pressure. *Ans.* Incidentally this indicates a head of $8 \times 2.3 = 18.4$ ft of water in the tank.

166. The flow of air. With certain modifications the principles described for the flow of water can be applied to the flow of air in ducts. The velocity, as for water, is dependent upon the square root of the velocity head and the acceleration of gravity; the quantity discharged upon the velocity and the area of the duct; the frictional losses being allowed for by coefficient or tabulated experimental data.

If the velocity head and the static head of moving air in a duct are determined by Pitot tube readings, see Art. 157, the *velocity* can be determined by equation (81), $v = \sqrt{2gh}$. It is customary to express air flow in feet per minute rather than per second. The Pitot tube is usually connected to a water manometer, the readings being in inches of water head. These can be reduced to equivalent feet head of air by the following proportion. Let h_{va} represent feet head of air, h_{vw} inches head of water (both velocity heads), w_w the weight of a cubic foot of water, and w_a the weight of a cubic foot of air at the conditions under consideration (as, for example, at 70 F and atmospheric pressure it will weigh

about 0.075 lb per cu ft). Then

$$\frac{h_{va}}{h_{vw} \div 12} = \frac{w_w}{w_a}$$

and

$$h_{va} = \frac{h_{vw}}{12} \times \frac{62.5}{w_a}$$

and the velocity in feet per minute is:

$$v = 60 \sqrt{\frac{2\,g \times h_{vw} \times 62.5}{12 \times w_a}} = 1096.5 \sqrt{\frac{h_w}{w_a}} \tag{87}$$

The theoretical *quantity discharged* in cubic feet per minute can then be found by the equation

$$Q = Av \tag{82}$$

The *air horsepower* can be found from the above quantity per minute and the total pressure in pounds per square foot. Let p represent total pressure in pounds per square feet, and h_p and h_v the pressure head and velocity head in inches of water as determined by the manometer.

Then $$p = (h_p + h_v) \times 0.0361 \times 144$$

Work in foot-pounds per minute equals pressure in pounds per square foot times area of duct in square feet times distance traveled in 1 minute (velocity in feet per minute), that is, in symbols $= p \times A \times v = p \times Q$ (since $A \times v = Q$).

Therefore $$\text{Horsepower} = \frac{p \times Q}{33{,}000} \tag{88}$$

Since Q = the quantity discharged in cubic feet per minute, it is sometimes thought of as the cubic-foot volume per minute, represented by V for volume; and then the formula is stated Horsepower = $pV/33{,}000$.

PROBLEMS

16–60. The barometer stands at 29 in. Compute atmospheric pressure in pounds per square inch.

16–61. Atmospheric pressure sustains 30.2 in. of mercury. How high a column of water will it sustain?

16–62. A pair of Magdeburg hemispheres are 6 in. in diameter. The pressure inside is 0.50 in. of mercury, outside 30 in. Compute the force required to pull them apart.

16–63. If a man's body has a surface of 18 sq ft, find the total pressure upon him in tons.

16–64. Find the greatest height that water can be carried over a siphon at the top of a mountain where the barometer stands at 26 in.

16–65. The piston of a steam engine has a diameter of 10 in., and the steam exerts a pressure upon it of 5 atmospheres. Find the effective force when the other side of the piston is exposed to the atmosphere.

16–66. What is the gauge pressure corresponding to a barometer reading of 29.2? What is the barometer reading if the atmospheric pressure is 14.3 lb per sq in.?

16–67. If a cylinder, 14 ft in diameter, is sealed when the barometer reads 30.5 in., what will the force on the head be when the barometer reads 28 in.? In which direction will this force act?

16–68. A gas inclosed in a cylinder with a freely moving piston has a volume of 1.23 cu ft when the barometer is at 29.5 in. To what height must the air pressure fall in order that the volume of the gas may become 1.34 cu ft at the same temperature?

16–69. What fractional part of the air in the receiver of an air-pump has been removed when the pressure within has fallen from 30 to 5.6 in.?

16–70. If a room contains 23 lb of air at a pressure of 15 lb, how many pounds will it contain at a pressure of 14.2 lb per sq in.?

16–71. The mouth of a vertical cylinder 18 in. high filled with air is closed by piston weighing 6 lb and area 6 sq in. If the piston is allowed to fall, how far will it descend?

16–72. 100 cu ft of air at pressure of 15 lb per sq in. is compressed to 36 cu ft at the same temperature. What will then be its pressure?

16–73. Plot a pressure-volume diagram from the following data, using pressure vertical and volume horizontal.

Pressure, Lb per Sq In.	Volumes, Cu In.	Pressures, Lb per Sq In.	Volumes, Cu In.
200	100	100	200
180	111	80	250
160	125	60	334
140	143	40	500
120	167	20	1000

16–74. A cylinder 40 in. long inside is fitted with a tight piston. When the gas inside is at atmospheric pressure, the inner face of the piston is 35 in. from the close end. The piston is pushed in so that the space within is successively 30, 25, 15, 10, and 5 in. Calculate the pressures at these positions

and plot the results on a pressure-volume diagram. (Plot volumes on the horizontal.)

16–75. If a gas in a cylinder at atmospheric pressure is compressed to $\frac{1}{8}$ its original volume, what will be the resulting pressure?

16–76. In a water-supply pressure tank the water stands 20 in. from the top, and the pressure gauge reads 4 lb. If water is pumped into the tank until the water level is 10 in. from the top, what will be the pressure gauge reading?

16–77. Same as Problem 16–76 if the water level stands 7 in. from the top.

16–78. The pressure of gas in a main is found to cause the water in one arm of an open-arm manometer to stand 2.4 in. higher than in the other. Find the pressure of the gas in pounds per square inch.

16–79. An open-arm manometer is filled with mercury and when it is connected to a certain cylinder of gas the mercury in the open arm rises to a position 2 in. above the level of the mercury in the closed arm. Find the pressure indicated in pounds per square inch.

16–80. I wish to measure the water pressure at a tap and connect with an open mercury manometer. When the tap is opened, the mercury is forced to a height of 42.5 in. in the open arm. Express the head in feet.

16–81. Determine the pressure indicated by the following closed manometer readings. Under atmospheric conditions: Air column = 10 in., mercury column = 7 in. (in open arm). Under pressure: Air column = 8 in., mercury column = 3 in.

16–82. Referring to the closed-arm manometer in the example in Art. 165, find the pressure and water level in the tank if the movement of mercury column had been only 1 in.

SUMMARY

Solids have elasticity of both shape and volume.

Liquids have elasticity of volume only.

Gases have elasticity of neither shape nor volume.

Density is weight per unit volume.

$$\rho = \frac{W}{V} \tag{79}$$

Specific gravity is the weight of a substance compared with an equal volume of water.

$$\text{Specific gravity} = \frac{\rho \text{ of substance}}{\rho \text{ of water}}$$

Pressure expresses the intensity of pressure per unit area, the unit being commonly pounds per square inch. Pressure depends upon the depth of liquid, called the **head**, and the density. 1 ft head of water

corresponds to a pressure of 0.434 lb per sq in.

$$p = wh \qquad \text{and} \quad h = \frac{p}{w} \tag{80a}$$

$$p = 0.434\,h \quad \text{and} \quad h = 2.304\,p \tag{80b}$$

Pascal's Principle states that pressure on any part of a fluid is transmitted equally on equal areas in all directions through the fluid.

The total pressure (force) on any plane surface is equal to the area in the liquid times vertical depth (head) from the surface of the liquid to the center of gravity of the area times the weight of a unit volume (density) of the liquid.

The center of pressure is the point of action of the single force equivalent to the total pressure of the liquid on a plane surface. For a surface that is not horizontal it is below the center of gravity; its location depends on the shape and position of the surface.

Buoyancy. Archimedes' Principle is that a body immersed in a fluid loses weight equal to the weight of the fluid displaced. The buoyant force on the body is equal to the weight lost.

The theoretical velocity (friction neglected) of flow from an orifice is the same as that acquired by a body having fallen from rest through a height equal to the head of water on the orifice.

$$v = \sqrt{2gh} \quad \text{and} \quad h = \frac{v^2}{2g} \tag{81}$$

The theoretical discharge is

$$Q = Av \tag{82}$$

The actual discharge is the theoretical amount modified by the coefficient of discharge, which is commonly taken as 0.61, being the product of the coefficients of contraction and velocity.

$$Q = 0.61\,A\sqrt{2gh} \tag{83}$$

Bernouilli's Theorem is that at any section of a tube or pipe, under steady flow without friction, the pressure head + the velocity head = the hydrostatic head.

$$H_1 = h_1 + \frac{v_1^2}{2g} \quad \text{and} \quad H_2 = h_2 + \frac{v_2^2}{2g} \tag{84}$$

The laws of friction in pipes are stated in Art. 158.

The relations of hydrostatic head, velocity head, friction head, and elevation are given by the equation:

$$H = h_e + h_p + h_v + h_f \tag{85}$$

Atmospheric pressure will support a mercury column in a barometer to a height of about 30 in. at sea level. **Standard pressure** is 30 in. of

mercury = 14.7 lb per sq in. For most practical purposes a pressure of 15 lb per sq in. is sufficiently exact.

Gauge pressure is stated without including that of the atmosphere.

Absolute pressure = gauge pressure + atmospheric pressure.

Boyle's Law is: The absolute pressure of a gas varies inversely as its volume, when the temperature remains constant.

$$\frac{p_1}{p_2} = \frac{V_2}{V_1} \quad \text{or} \quad p_1 V_1 = p_2 V_2 \tag{86a}$$

Dalton's Law is: When a mixture of several gases at the same temperature is contained in a vessel, each gas produces the same pressure as if the others were not present. The total pressure in the container equals the sum of the several pressures.

Pressure gauges and manometers are used to measure pressure. A vacuum gauge is graduated in inches of mercury, 30 in. of mercury corresponding to a perfect vacuum = 0 absolute pressure and −15 lb (or 14.7) gauge pressure. Open-arm manometers show the pressure by the difference in level of the two arms. Mercury for average pressures, and water for low pressures, are used. For higher pressures the closed-arm manometer is used. The pressure is computed by applying Boyle's Law to the volume of air trapped in the closed arm.

Air flow, with some modifications, can be computed in the manner of water flow. With proper units the equations are:

$$v = \sqrt{2gh} = 1096.5\sqrt{\frac{h_w}{w_a}} \tag{87}$$

$$\text{Horsepower} = \frac{(p \times Q)}{33{,}000} \quad \text{or} \quad \frac{pV}{33{,}000} \tag{88}$$

CHAPTER XVII

ELEMENTS OF HEAT

The material in this chapter, although ordinarily not found in textbooks on mechanics, is included because of the importance of heat energy to numerous practical applications of applied mechanics. Both subjects, mechanics and heat, are applied physics. Both are required in the first year studies of many industrial schools.

Modern engineering practice employs thermal units in the rating of power plants and related mechanical processes. A knowledge of the terms of heat engineering is necessary to understand the significance of such terms as the thermal efficiency of steam- and hydroelectric power units.

Only the fundamental principles are treated in this book, with some reference to a few of the important applications. For a complete discussion of steam engines, gas engines, heating plants, refrigeration, and air conditioning, reference should be made to some textbook on industrial heat.

167. Molecular energy. Heat is a form of energy due to molecular action. The molecules of matter which compose a body are in constant vibration, striking against one another and causing the sensation of heat. The more rapid the vibration, the greater is the amount of heat, and the hotter is the body. This is true whether the body is a solid or a fluid.

The vibration is due to the transfer of some form of energy to the body, as, for example, the rubbing of a dry shaft in an unlubricated bearing, or the striking of a nail repeatedly with a hammer. If the object has more heat energy than one's hand, it feels hot to the hand; if not so much heat energy, it feels cold. But cold is simply the lack of heat, and there is always some heat in a body so long as there is any molecular action. Molecular action is believed to cease at the very low temperature of 460 deg below zero, fahrenheit. This temperature level is called the absolute zero.

Heat is one form of energy. Energy can be transformed into different forms, as mechanical or electrical or chemical energy, etc. As has been stated previously (Art. 91) the Law of the Con-

servation of Energy is that no energy is ever destroyed or created. It may be transformed and must often be wasted during its transfer for some useful purpose, but a careful analysis shows that none is actually destroyed.

Matter occurs in three physical states, each kind of material normally in only one state, i.e., either solid, liquid, or gaseous. However, the state of most substances can be altered by the addition or removal of heat energy. A change in state also may be accomplished by a change in pressure. A familiar example is that of water, which occurs as a solid, liquid, or gas, under varying conditions of temperature or pressure.

168. Intensity of heat. Temperature. The intensity of heat in a body is expressed by stating its temperature. Temperature is a measure of the intensity of the molecular vibrations. **The unit of intensity is the degree, as measured on a temperature scale.** The **fahrenheit scale** is most commonly used by engineers.

Thermometers are used to measure temperature. For very high temperatures, beyond the range of a thermometer, some form of pyrometer is used. The thermometer consists of a glass tube, closed at the upper end, with a bulb at the lower end containing some liquid, usually mercury. When the mercury is heated it expands and moves up the stem of the tube. The positions of the mercury at the temperatures of melting ice and of boiling water are marked on the stem and, on the *fahrenheit* thermometer, the distance between is divided into 180 equal parts called degrees. The freezing, or melting, point is called 32 deg and the boiling point is therefore 212 deg.

In the **centigrade thermometer scale,** used in some electrical and scientific operations, the freezing point is marked 0 and the boiling point 100. Each centigrade degree is, therefore, larger than a fahrenheit degree in the ratio of 180 to 100 = 9 to 5. Also the centigrade zero is at a higher temperature than the fahrenheit zero since 0 C = 32 F. Therefore to change from the centigrade scale to the fahrenheit, it is necessary to multiply the degrees centigrade by $\frac{9}{5}$ and add 32 deg. Letting F represent the temperature in degrees fahrenheit, and C the temperature in degrees centigrade, the relation between the two scales may be expressed by the following equations:

$$F = (C \times \tfrac{9}{5}) + 32 \qquad (89a)$$

and

$$C = (F - 32) \times \tfrac{5}{9} \qquad (89b)$$

For computations involving certain laws of fluids it will be necessary to employ the *absolute scale of temperature* in which the zero is 460 deg below 0 F. Letting T represent the absolute temperature, and t the ordinary fahrenheit temperature, the relation is:

$$T = t + 460 \tag{90}$$

Heat which is apparent as a change of temperature is called **sensible heat,** to distinguish it from latent heat which is stored in a substance due to change of state. See Art. 174.

Example.

Express a room temperature of 68 F in the centigrade scale; also in the absolute scale.

Solution.

$$C = (F - 32) \times \tfrac{5}{9} = (68 - 32) \times \tfrac{5}{9} = 20\ C \quad \textit{Ans.}$$

$$T = t + 460 = 68 + 460 = 528 \text{ degrees abs.} \quad \textit{Ans.}$$

169. Expansion and contraction due to temperature changes. The property by which a body increases in length when its temperature is raised is common to practically all materials, with a few exceptions. In solids the *linear expansion* is of principal importance, whereas in fluids *cubic expansion,* in all directions, must be considered.

There are many familiar illustrations of temperature expansion and contraction. It is made use of in the thermometer, and also in the steam engine. A steam pipe becomes longer when heated, and, if long enough, an *expansion joint* must be installed. Railroad rails expand in the summer, making the space between ends less than in the winter. Rails welded end to end, as in many trolley lines, are not free to expand lengthwise, nor to bend with the tendency to expand because of being anchored in concrete base or by the ties and cross rods. They are, therefore, under stress as noted in Art. 122. Buildings have sliding expansion joints in the steel construction. A bridge truss has a roller or rocker expansion joint on the supporting abutment at one end. A steel tape, unless made of invar or some other alloy with a low expansion rate, will give a difference of several feet in the measurement of a mile in summer and in winter; which of course is corrected in an accurate survey.

The coefficient of expansion is the rate of expansion per unit length per degree rise. It is quite constant for each material.

For example, the coefficient of linear expansion, C, for steel is 0.0000065 which indicates that if the temperature of a steel rod 1 ft long is raised 1 F its length will increase 0.0000065 ft; and a piece 1 in. long would increase the same proportion of an inch. The coefficient of cubic expansion of a substance is about three times the linear coefficient. The linear coefficients of several materials are tabulated in the Appendix.

Example.

If a steel rail measures exactly 40 ft in winter at a temperature of 20 F, what will be its length at a temperature of 100 F?

Solution.

Expansion of 1 ft for 1 deg change = 0.0000065 ft

Expansion for conditions given

$= 40 \times (100 - 20) \times 0.0000065 = 0.02$ ft (= about $\frac{1}{4}$ in.)

Length $= 40 + 0.02 = 40.02$ ft *Ans.*

PROBLEMS

17–1. Reduce 50 F to centigrade and 50 C to fahrenheit.

17–2. The boiling points of alcohol and turpentine are given in tables as 78 C and 156 C, respectively. Reduce these to fahrenheit.

17–3. Find the equivalents for the following: −4 C; 20 C; 39 F; 62 F; 72 F.

17–4. Compute the number of degrees fahrenheit difference between an outdoor temperature 50 C below the centigrade zero and an indoor temperature of 65 F.

17–5. If an iron pipe line, 80 ft long, varies in temperature from 42 F to 200 F, what will be the increase in its length?

17–6. A run of steel pipe measures 64 ft 3 in. when steam at 212 F is passing through it. What would be its length if water at temperature of melting ice were running through it?

17–7. Find the expansion in length of a steel I beam, 18 ft long, if the temperature change is 150 deg.

17–8. A run of brass pipe measures 114.46 ft at a temperature of 196 F. Find its length at a temperature of 40 F.

170. Expansion of gases. Laws of gases. All gases expand with increase of temperature at practically the same rate, provided the pressure does not change. This rate is about $\frac{1}{492}$ of the volume for each degree fahrenheit. If the gas is confined, so that the volume remains constant, then the pressure will change, increasing with the temperature at about the same rate.

A consideration of this shows that with pressure constant, a gas will expand or contract $\frac{1}{492}$ for each degree change with the result that at 492 deg below freezing, or at -460 F, there would be a zero volume. For certain mathematical calculations it is proper to assume such a possibility and to call the temperature at which it would occur the *absolute zero* of temperature. In the absolute scale the freezing point of water is 460 deg + 32 deg = 492 deg absolute.

Charles' Law is that the volume of a gas is directly proportional to its absolute temperature, provided the pressure remains constant, i.e.,

$$\frac{V_1}{V_2} = \frac{T_1}{T_2} \tag{91}$$

If a cubic foot of air at 40 F is heated to 100 F, with no change in pressure, its new volume will be to 1 cu ft as (100 + 460) is to (40 + 460). That is, it will equal $1 \times \frac{560}{500} = 1.12$ cu ft.

Further, **the absolute pressure of a gas is directly proportional to its absolute temperature, provided the volume remains constant,** i.e.

$$\frac{p_1}{p_2} = \frac{T_1}{T_2} \tag{92}$$

If air in a tight container at 15 lb absolute pressure is heated from 40 F to 100 F, the volume will remain constant, and the new pressure will be to 15 as 560 is to 500; or it will equal $15 \times \frac{560}{500} = 16.8$ lb per sq in. absolute.

Boyle's Law (see Art. 164) **is that the absolute pressure of a gas is inversely proportional to its volume provided the temperature remains constant,** i.e.,

$$\frac{p_1}{p_2} = \frac{V_2}{V_1} \tag{86a}$$

In practice many cases occur in which none of the three conditions remain constant and a combination of the above laws is necessary, giving the **general law of a perfect gas:**

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2} \tag{93}$$

in which p and T are always in absolute units.

The general law can be derived from (91) and (86*a*) by combining them thus:

$$\frac{V_1}{V_2} = \frac{T_1}{T_2} \times \frac{p_2}{p_1}$$

whence

$$\frac{V_1}{V_2} \times \frac{p_1}{p_2} = \frac{T_1}{T_2}$$

and

$$\frac{V_1 p_1}{T_1} = \frac{V_2 p_2}{T_2}$$

PROBLEMS

17–9. 1000 cu ft of dry air is heated from 0 F to 70 F; how much does it expand?

17–10. Gas in a cubic foot container at atmospheric pressure is heated from 0 F to 70 F. Find the new pressure.

17–11. A cylinder 6 in. in diameter and 2 ft long is filled with a gas at atmospheric pressure. If the cylinder is shortened to 1 ft length and the temperature increased from 50 F to 100 F, what will be the pressure?

17–12. Six cubic feet of a gas at 68 F is heated to 188 F. Find its new volume if the pressure remains constant.

17–13. A quantity of gas in a container is heated from 60 F to 100 F. Find the pressure.

17–14. If .85 cu ft of gas is heated from 70 F to 110 F, and the pressure is increased from 0 to 3 lb on the gauge, what will be the resulting volume?

171. Quantity of heat. The British thermal unit (abbreviated Btu) is the unit of quantity. It is the quantity of heat necessary to raise one pound of water one degree fahrenheit.

Quantity differs from intensity in that it takes into account both temperature and weight of the substance. A large volume, or weight, of water at any given temperature contains a greater quantity of heat than does a small volume of water. Since 1 Btu is required to raise a pound of water 1 deg, 2 Btu will be required to raise its temperature 2 deg. Likewise, 2 Btu will be required to raise 2 lb of water 1 deg. When cooling, the same quantity of heat is released as is absorbed when heating.

If H represents the quantity of heat in Btu, W the weight in pounds, t_1 the initial temperature, and t_2 the final temperature, the relation for water only is expressed thus:

$$H = W\,(t_2 - t_1) \qquad (94)$$

The equation for *any* material will be treated in Art. 172.

Example.

Find the heat given off by the cooling of 100 lb of water from 120 F to 70 F.

Solution.

$$H = W(t_2 - t_1) = 100\ (120 - 70) = 5000 \text{ Btu} \quad \textit{Ans.}$$

Note. In the metric system the weight is expressed in grams, the temperature in centigrade degrees, and the heat units (grams times degrees centigrade) in *calories*. 1 Btu is equivalent to about 252 calories.

172. Specific heat. The quantity of heat necessary to raise the temperature of any substance any given amount, varies according to the specific heat of the substance. The specific heat of water is considered unity.

The specific heat of a substance is the number of British thermal units necessary to raise the temperature of one pound of the substance one degree fahrenheit. Since the specific heat of water is 1, the specific heat of any substance is the number which expresses the *ratio* of the quantity of heat required to raise the temperature of the substance *to the quantity required to raise the same weight of water through the same change in temperature.*

For example, the specific heat of lead is about 0.031, which indicates that 0.031 Btu will cause a temperature rise of 1 deg in 1 lb of lead; and that this is 0.031 as much heat as is required for an equal weight of water. Conversely, if the same amount of heat is added to a pound of lead and a pound of water, the temperature rise of the water will be only 0.031 times as great as that of the lead. If the heat added is 1 Btu, the water will rise 1 deg and the lead about 33 deg.

The equation of the preceding article can now be written for *any* material, letting c represent the specific heat, thus:

$$H = cW(t_2 - t_1) \tag{95}$$

Example.

What will be the temperature rise in 5 lb of lead due to the addition of 10 Btu?

Solution.

$$H = cW(t_2 - t_1)$$

$$10 = 0.031 \times 5 \times (t_2 - t_1)$$

$$(t_2 - t_1) = \frac{10}{0.031 \times 5} = 64.5 \text{ deg rise} \quad \textit{Ans.}$$

When considering the specific heat of a gas attention must be given to the fact that an increase in temperature causes an expansion of volume, or an increase in pressure if the gas is in a container which prevents its expansion. Thus it is necessary to know the conditions because the specific heat of a gas is higher when the pressure is kept constant than when the volume remains constant. *The average values for air are:*

Specific heat at constant volume = C_v = 0.169 Btu.
Specific heat at constant pressure = c_p = 0.238 Btu.
(The latter value is commonly used as 0.24.)

The specific heats of some materials are listed in the Appendix.

173. Temperature of mixtures. The heat equation. When two or more substances at different temperatures are mixed together, or are brought into contact, heat energy is transferred from the hotter to the cooler substance until both are at the same temperature. The law covering the conditions of this transfer of energy is called the heat equation. **The heat equation is:**

Heat lost (by hot body) = heat gained (by cold body).

This means, for example, that if hot water is poured into a container of cold water, resulting in a temperature of the mixture higher than that of the original cold water and lower than that of the hot water, the number of British thermal units lost by the hot water equals that gained by the cold. The heat lost, in British thermal units, equals the weight of the hot water times its temperature drop; the heat gained equals the weight of the cold water times its temperature rise.

Due allowance must be made for loss by radiation into the air, and into the container of the cold water. The radiation into the air can be neglected in many approximate computations. More accurate results will be obtained by the use of insulated containers called calorimeters, which decrease the radiation to a minimum.

The loss into the container is often computed by the use of its **water equivalent,** which is the weight of water that would absorb an equal quantity of heat units. **The water equivalent is equal to the specific heat of the material multiplied by its weight.** For example, the specific heat of copper is approximately 0.1, and since therefore, copper absorbs heat $\frac{1}{10}$ as rapidly as water, $\frac{1}{10}$ the

weight of the container is added to the weight of the cold water as its water equivalent.

If some substance other than water is used, as if a block of iron is heated and then cooled in water, or if it is cooled in oil, the specific heat of each substance must be included as described in the preceding article.

Let W_c be the weight of the cold body, t_c its temperature, and c_c its specific heat; also let W_h, t_h, and c_h represent the corresponding values for the hot body; let t_m be the final temperature of the mixture: Then the heat equation becomes:

$$c_h W_h (t_h - t_m) = c_c W_c (t_m - t_c) \tag{96}$$

Another method, which is more convenient when several materials are involved, and when it is not easy to foresee whether the temperature of the mixture will be above or below that of certain of the bodies, is known as the **method of total heat.** An equation may be written expressing the equality between the total heat* of all the bodies, above an arbitrary base temperature level, *before* and *after* the exchange of heat. The base is chosen below the lowest original temperature of any material in the particular case. It is commonly taken as 32 F; also sometimes as 0 F, if it is evident no difficulty will be encountered by going below the freezing point.

Suppose that three bodies have weights W_1, W_2, and W_3; with specific heats c_1, c_2, and c_3; and temperatures t_1, t_2, and t_3; and suppose that after mixture the temperature is t_m, and that the base used is t_b.

Then the total heat *above the base* before mixing equals the total heat after mixing, that is,

$$\begin{aligned} c_1 W_1 (t_1 - t_b) + c_2 W_2 (t_2 - t_b) + c_3 W_3 (t_3 - t_b) = \\ c_1 W_1 (t_m - t_b) + c_2 W_2 (t_m - t_b) + c_3 W_3 (t_m - t_b) \end{aligned} \tag{97}$$

In some calorimetric processes, the value of t_m is determined, and then the specific heat of one of the bodies may be computed from the equation.

The specific heat of some materials is given in the Appendix.

* The term "total heat" is not strictly accurate, as it means only the amount above an arbitrary base, whereas there is heat also below this base at any temperature above absolute zero.

Example 1.

In a laboratory experiment a piece of cast iron weighing 2 lb was immersed in 3 lb of water at 60 F contained in a copper container weighing 1 lb. If the temperature of the iron was 210 F, compute the final temperature of the three bodies after mixing, neglecting radiation.

Solution.

Wt of cold water + water equivalent of container =

$3 + (\frac{1}{10} \times 1) = 3.1$ lb

$c_h W_h (t_h - t_m) = c_c W_c (t_m - t_c)$

$(.130)\ (2)\ (210 - t_m) = (1)\ (3.1)\ (t_m - 60)$

$54.6 - 0.26\ t_m = 3.1\ t_m - 186$

$t_m = 71.6$ F *Ans.*

Example 2.

Solve Example 1 by the method of total heat above 32 F.

Solution.

$(.130)\ (2)\ (210 - 32) + (1)\ (3)\ (60 - 32) + (0.1)\ (1)\ (60 - 32)$

$= (.130)\ (2)\ (t_m - 32) + (1)\ (3)\ (t_m - 32) + (0.1)\ (1)\ (t_m - 32)$

$46.8 + 84 + 2.8 = 0.26\ t_m - 8.32 + 3\ t_m - 96 + 0.1\ t_m - 3.2$

$3.36\ t_m = 240.6$

$t_m = 71.6$ F *Ans.*

Example 3.

Since all temperatures are above the freezing point, Example 1 may be solved by the method of total heat above 0 F as a base. It is mathematically simpler than the preceding solution.

Note that this expression refers strictly to the *sensible* heat above 0 F and neglects the latent heat of fusion of ice.

Solution.

$(0.13)\ (2)\ (210) + (1)\ (3)\ (60) + (0.1)\ (1)\ (60) =$

$(0.130)\ (2)\ (t_m) + (1)\ (3)\ (t_m) + (0.1)\ (1)\ (t_m)$

$3.36\ t_m = 240.6$

$t_m = 71.6$ F *Ans.*

PROBLEMS

17–15. Find the number of British thermal units required to raise 1 cu ft of water from freezing to boiling point.

17–16. Find the number of British thermal units required to raise 60 gal water from 62 F to 100 F.

17–17. If 600 Btu are supplied to 30 lb of water at 68 F, what will be the resulting temperature?

17–18. Two hundred and fifteen lb of water at a temperature of 140 F cool to 72 F. How much heat is released?

17–19. A hospital housing 150 persons requires 35 gal of water at 140 F, per day per person. Compute the total British thermal units per day to be delivered to this water based on an average city water temperature of 52 F.

17–20. Two lb of water at 210 F are mixed with 7 lb of water at 39 F. Find the resulting temperature, making no allowance for radiation.

17–21. How much water at 212 F must be mixed with 10 lb at 39 F to give a temperature of 140 F?

17–22. If 10 gal of water at 145 F are mixed with 20 gal at 60 F, what will be the resulting temperature assuming there is no radiation?

17–23. Find the number of British thermal units required to heat 10 lb of iron from 50 F to 210 F; from 210 F to 2500 F.

17–24. If 50 lb of lead receive 50 Btu of heat, what will be the temperature rise?

17–25. If 10 lb of lead are to be heated from 50 F to 621 F (melting point), how much heat will be required?

17–26. A machine shop contains steel machinery weighing 70,000 lb. The room temperature and that of the machines stands at 40 F when heat is turned on. Compute the number of British thermal units absorbed by the machinery before it reaches 70 F.

17–27. If 6 lb of lead at 300 F are placed in 2 lb of water at 70 F, what will be the resulting temperature?

17–28. A body weighing 4 lb whose temperature is 190 F is placed in 4 lb of water at 35 F. The resulting temperature is 45 F. Find the specific heat of the body.

17–29. If 3 lb of lead at 200 F, and 4 lb of iron at 190 F are put into 10 lb of water at 60 F, what will be the resulting temperature?

17–30. A sample of fuel weighing 0.02 lb is burned to an ash in a calorimeter, heating 13 lb of water from 62 F to 84 F. The calorimeter includes 2.3 lb of brass and 0.1 lb of glass which are heated the same as the water. Find the heat value of the fuel in British thermal units per pound of fuel.

174. Latent heat. Latent heat is that which is stored up in a body when there is a change of state from solid to liquid, or from liquid to gaseous. This heat is released by the reverse changes.

Latent heat causes or accompanies a change of state, but not a change in temperature. Heat which causes or accompanies a change in temperature is called **sensible heat.**

When a solid is heated sufficiently it begins to change to a liquid, this change being called fusion. The temperature at which fusion, or melting, takes place is called the melting point or the

temperature of fusion. Further addition of heat does not raise the temperature until the substance is entirely liquefied. The amount of heat absorbed and stored during fusion varies with the material. To melt 1 lb of ice at 32 F, 144 Btu are required, and this figure is called the **latent heat of fusion of ice.**

When a liquid is heated until the temperature has reached the boiling point, or temperature of vaporization, the change to a gas begins. The temperature then continues constant and the heat is absorbed by the gas as latent heat. When vaporization is complete the further addition of heat will result in further temperature rise. The vaporization point of water is 212 F and the **latent heat of vaporization of water** is 970 Btu per lb.

The condensation of steam releases the same amount of latent heat.

Example 1.

How much heat will be necessary (*a*) to melt 22 lb of ice? (*b*) to vaporize 22 lb of water?

Solution.

(*a*) Heat = weight $\times$ latent heat = $22 \times 144 = 3168$ Btu. *Ans.*
(*b*) Heat = $22 \times 970 = 21{,}340$ Btu. *Ans.*

Example 2.

How much heat will be released by the condensation of 9 lb of steam?

Solution.

Heat = weight $\times$ latent heat = $9 \times 970 = 8730$ Btu. *Ans.*

175. Temperature changes in water. Water occurs in three states; solid, liquid, and gas. The changes from one state to another are caused or accompanied by changes in temperature and in heat content. Starting with 1 lb of ice at any selected temperature, as 0 F, the addition of 16 Btu will result in a temperature of 32 F. This is because the specific heat of ice is about 0:5, and the heat required = $H = cW\ (t_2 - t_1) = 0.5 \times 1 \times (32 - 0) = 16$ Btu. The further addition of 144 Btu will furnish sufficient latent heat to melt the ice into water at 32 F, still 1 lb in weight. The further addition of 180 Btu will raise the temperature to 212 F. The addition of 970 Btu more will evaporate the pound of water into 1 lb of steam at 212 F. And each half British thermal unit added further will raise the steam 1 deg

(since the specific heat of steam under atmospheric pressure is about 0.5). These facts can be tabulated as follows:

Ht. Added Btu.	Sp. Ht.	Total Ht. Btu.	Temp. (F)	Wgt. (lb)	State
...	...	...	0°	1	Solid — Ice
16	0.5	16	32	1	Solid — Ice
144	...	160	32	1	Liquid — Water
180	1.0	340	212	1	Liquid — Water
970	...	1310	212	1	Gas — Steam
0.5	0.5	1310.5	213	1	Gas — Steam

While the above changes take place the volume will change. If the volume were not free to change, then a change in pressure would result, causing a change in the melting and boiling points. The pound of water will contract in volume as its temperature rises from 32 F to 39 F, which is its *point of maximum density* and above which it begins to expand. Almost all materials expand with increased temperature, and water follows this rule above 39 F. But the fact that *it is an exception to the rule below* 39 F makes it lighter at the freezing point, with the result that ice forms at the surface instead of at the bottom of a reservoir or lake. For the exact relations of temperature, density, and volume, reference must be made to tables; the approximate increase in volume between 39 F and 212 F is about 4 per cent. When water vaporizes there is a very great change in volume, approximately 1600 times. A convenient way to remember is that a cubic *inch* of water will make nearly a cubic *foot* of steam, if allowed to expand freely at atmospheric pressure.

PROBLEMS

17–31. How much heat will be required to vaporize 30 gal of water at 212 F?

17–32. How many British thermal units will be absorbed by the melting of 100 lb of ice at 22 F into water which is then warmed to 58 F?

17–33. How many pounds of ice at 32 F must be melted to water at 32 F to give an absorption of 5000 Btu?

17–34. Compute the number of British thermal units required to raise the temperature of 100 lb of ice from 20 F to 32 F, melt it, raise the temperature to 212 F, vaporize all the water at atmospheric pressure, and superheat the vapor through 10 deg.

17–35. If 2000 Btu are added to 3 lb of water at 170 F, how much water will be vaporized? (The entire quantity must be heated to boiling point, before any evaporates.)

17–36. How many British thermal units must be removed from 100 lb of water at 60 F in order to produce a block of ice of the same weight at 5 F?

17–37. Compute the amount of heat per hour absorbed by the water in the cooling coil of an air-conditioning unit, if the water is 46 F at entrance and 50 F at the outlet, and if the rate of flow is 5 gpm.

17–38. If 200 gal of water per hour are circulated through the coil of a heating unit, entering at 180 F and leaving at 90 F find the number of British thermal units released to the coil.

17–39. Find the required number of gallons of water entering a cooling coil at 46 F and leaving at 56 F in order to absorb 10,000 Btu.

17–40. Compute the weight of ice required to absorb 10,000 Btu if the temperature change is from 10 F to 60 F.

176. Fuels and thermal efficiency. Heat energy is furnished by the combustion of a fuel. The chemical energy stored in the fuel is transformed into the form of heat. A portion of this can be employed for useful purposes, while a portion must necessarily be wasted in the process.

The number of British thermal units released by the complete combustion of a fuel can be determined by calorimetry. (Art. 173.) Some fuel values are as follows:

Btu's of Heat Produced by Combustion of One Pound of Fuel

Acetylene	21,400	Hard Coal	15,000
Hydrogen	62,000	Soft Coal	14,000
Fuel Oil	18,600	Coke	13,000
Gasoline	20,300	Wood	7,000
Kerosene	20,000	Carbon	14,400
Illuminating Gas, per cubic foot			600

The equivalent units previously discussed are tabulated here for reference.

$$1 \text{ Btu} = 778 \text{ ft-lb}$$

$$1 \text{ Btu} = 1055 \text{ w-sec}$$

$$1 \text{ whr} = 3.4 \text{ Btu}$$

A watthour is the energy equivalent to 1 watt of power used for 1 hour of time, as 2 watts times $\frac{1}{2}$ hr or $\frac{1}{2}$ watt times 2 hr, etc.

Thermal efficiency is the ratio of the output in thermal units to the input in thermal units. Like mechanical efficiency, it is expressed in percentage.

Example.

Find the thermal efficiency of a small gas burner if it consumes 10 cu ft of gas in order to heat 20 lb of water through 40 deg.

Solution.

$$\text{Output} = 20 \text{ lb} \times 40 \text{ deg} = 800 \text{ Btu}$$

$$\text{Input} = 10 \text{ cu ft} \times 600 \text{ Btu per cu ft} = 6000 \text{ Btu}$$

$$\text{Thermal efficiency} = \frac{\text{output}}{\text{input}} = \frac{800}{6000} = 0.13 = 13 \text{ per cent } \textit{Ans.}$$

PROBLEMS

17–41. A hot-water heater consumes 125 cu ft of gas for each 30 gal of water heated from 58 F to 140 F. Compute the thermal efficiency of the heater in percentage.

17–42. An insulated hot-water storage heater is rated as 85 per cent efficient. If the capacity of the tank is 50 gal, and the temperature of the entering city water is 48 F, how much gas will be required to heat the whole tankful to a temperature of 120 F?

17–43. If a small light and power unit consumes 12 oz of fuel in 60 min and delivers 16 amperes at 32 volts over the same period, find (*a*) the input in British thermal units, (*b*) the output in British thermal units (equivalent to electrical wattage), (*c*) the over-all thermal efficiency of the set. Allow 20,000 Btu per lb of fuel.

177. Heat transfer. Heat moves from any temperature level to a lower one. *Conduction* is the process of heat transfer in which the molecular energy is passed from one molecule to an adjoining one of lower temperature by contact; and similarly from one body to a cooler one by contact. An illustration is that of the heat traveling along a rod, one end of which is held in a fire. The rapidity of transfer by conduction varies in different materials. Solids are much better conductors than liquids, and liquids are better than gases. Among the solids, metals are better than nonmetals, silver being the best. The poor conductors are called insulators, of which cork and rockwool are examples.

Convection currents are caused when the substance heated is a liquid or a gas. The hot particles expand and become lighter than the cooler ones, and rise as the cool particles descend, thus distributing the heat throughout the whole body. Convection currents are useful in warm-air and hot-water heating systems, in ventilation systems, etc.

Radiation is the process of heat transfer when a body is placed so as to intercept the heat rays from a hot object. The sun's

heat is brought to the earth by radiation. An open fire, or an electric "sun bowl" throw out radiant heat. Radiant heat travels in waves in straight lines, as does light, through a vacuum, or through air or glass without transferring its energy until intercepted by some substance through which the waves cannot pass, which then receives the heat. The quantity transmitted by radiation is relatively small; the warmth is noticeable if in front of a fireplace or radiator, but the heating of a room cannot be accomplished to any extent by radiation alone. It is necessary to warm the air by conduction and distribute it by convection.

Radiant heat can be reflected by light surfaces and absorbed by dull ones in the same manner as light rays. A dull surface absorbs heat readily, and also transmits it readily, and is therefore better for a radiator surface for heating a room than is a bright surface which reflects the rays and does not transmit them so readily. A bright-surfaced water tank is better for storing hot water. Modern hot-water tanks are built with an insulated casing, as, for example, some insulating fibre or cork placed loosely between an inner and outer shell.

In practice the heat that can or will be transmitted through a body is computed by use of an experimentally determined coefficient called the **coefficient of heat transmission,** or of heat transfer. Its symbol is U. **U is the number of British thermal units transmitted, per square foot of surface, per degree difference in temperature (between the warm and cool sides), per hour of time.** The coefficient takes into consideration the structure and thickness of the material, the radiation from the cool side, the effect of the air film on both sides, and of air spaces in the substance. Its value is for the thickness of material as generally used.

Some values of U are:

Window Glass	1.13
Cork Insulation (2 in.)	.14
Brick Wall, 8 in., plain	.39
Brick Wall, 8 in., plastered	.36
Brick Wall, 12 in., plain	.30
Brick Wall, 12 in., plastered	.28
Framed Wall, plaster, lath, studding, sheathing, and clapboard siding	.26
Cast-iron Radiator	1.8
Steam Pipe, bare	2.2

For more precise computations *coefficients of heat conductivity* are employed. The symbol for these coefficients is k. k is the number of British thermal units that will be conducted *through a thickness of* 1 *in.* of the material, per square foot, per degree differential, per hour of time. By the use of different values of k for each kind of material in the wall, together with values for the air films inside and out and for air spaces, a value for the over-all coefficient U can be computed by formulas. That method will not be treated in this book.

In computing the requirements for heating a room, the loss through windows and walls must be estimated, and also through floor and ceiling if the rooms below and above are not heated, to determine the required number of British thermal units to be provided by the heating system to replace the losses. Heat must also be supplied to replace the loss by change in the air in the room which filters out around doors and windows. This is commonly taken as equivalent to one complete change per hour unless there is more rapid forced ventilation. Each 55 cu ft of air space requires about 1 Btu, per degree differential, per hour. This can be determined from the average density of a cubic foot of air ($= 0.075$ lb) and its average specific heat ($= 0.24$). Thus for each degree of temperature $H = cW = 0.24 \times 0.075 = 0.018$ Btu, i.e., about $\frac{1}{55}$ Btu per cu ft.

Heat transfer can be stated in the form of an equation, thus: let H represent the quantity in British thermal units transferred per hour, A the area in square feet, U the coefficient of heat transfer (British thermal units per square foot per degree difference per hour) for the material under consideration, t_i the temperature on the inside (or higher temperature), t_o the outside temperature (or lower temperature). Then:

$$H = AU\ (t_i - t_o) \tag{98}$$

Example 1.

Find the hourly heat loss through three windows 66 in. high and 30 in. wide, if the temperature inside is 70 F and outside is 0 F.

Solution.

$$H = 3 \times (5.5 \times 2.5) \times 1.13 \times (70 - 0) = 3263 \text{ Btu} \quad Ans.$$

Example 2.

Estimate the heating requirements for a corner room, 10 ft $\times$ 15 ft $\times$ 8 ft high, with two windows in each outside wall, 2.5 ft $\times$ 6 ft.

Outside temperature 0 F, inside 70 F. Assume heated rooms adjoining and above and below; frame construction.

Solution.

Volume of air space $= 10 \times 15 \times 8 = 1200$ cu ft
Area of glass $= 4 \times 2.5 \times 6 = 60$ sq ft
Area of outside walls (gross) $= (15 + 10) \times 8 = 200$ sq ft
Net wall area $= 200 - 60 = 140$ sq ft

Then, for each degree difference:

Heat for air change $= 1200 \times 0.018 = 21.6$ Btu
Heat for glass area $= 60 \times 1.13 = 67.8$ Btu
Heat for wall area $= 140 \times 0.26 = 36.4$ Btu
Total heat $= (21.6 + 67.8 + 36.4) \times (70 - 0) = 8806$ Btu. *Ans.*

Example 3.

Find the number of British thermal units that can be transmitted through 1 sq ft of a cast-iron radiator containing steam condensing at 212 F with a room temperature of 72 F.

Solution.

$$H = AU\,(t_t - t_o) = 1 \times 1.8 \times (212 - 72) = 252 \text{ Btu} \quad \textit{Ans.}$$

178. Laboratory determination of transfer coefficients. A method of measurement of heat transfer is furnished by the use of a " cork heat box," as illustrated in Fig. 261. The box is constructed of 2 in. cork board, about 3 ft $\times$ 2 ft $\times$ 2 ft in size. It is fitted with an electric heating coil and with thermometers to obtain the temperature at several points inside near the walls and ceiling. The quantity of heat put in can be regulated by a rheostat, and measured by voltmeter and ammeter readings.

In performing a test the object is to raise the temperature inside to about 70 deg above the outside (laboratory temperature). Care is taken to maintain a steady average differential over a period of time, preferably an hour. This requires considerable preliminary adjustment, to avoid overheating and to allow for the material to become saturated.

When a state of *heat balance* has been obtained and maintained for a sufficient period a heat equation can be stated. The heat put into the box equals the heat transmitted through the five walls of cork and one wall of material being tested, such as window glass, lath and plaster, wall board and other building and insulating materials. Preliminary to the test of a building material, several

tests are run with all six sides of cork board to determine the coefficient for the box.

The heat put in = watts $\times$ 3.4 Btu (for each hour of time).

The heat leaking out = (area of five sides of cork $\times$ the coefficient for the box $\times$ the temperature difference) + (area of the specimen $\times$ the temperature difference $\times$ the coefficient for the specimen). The last term is the unknown in the equation.

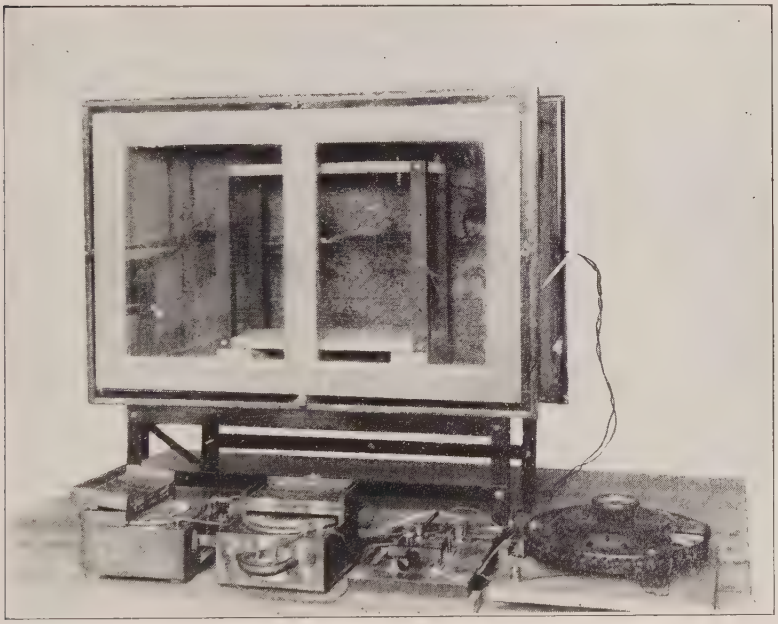


FIG. 261. Laboratory test-box for study of heat transfer.

PROBLEMS

17–44. If the temperature inside a room 14 ft $\times$ 12 ft $\times$ 8 ft high is 70 F and outside is 0 F, find the probable heat loss per hour through one end and one side wall of plaster on 8 in. brick, with 2 windows 5 ft $\times$ 3 ft.

17–45. Find the probable heat loss per hour through the exposed end and side walls of a corner room 14 ft $\times$ 22 ft $\times$ 8 ft high, and through 4 windows 2 ft 6 in. $\times$ 6 ft, if the temperature is 70 F inside and 0 F outside.

17–46. How much heat per hour should be supplied to a room 16 $\times$ 12 $\times$ 8.5 ft exposed on one side and two ends, frame construction, with four windows 6 $\times$ 3 ft, with one air change per hour, to keep the temperature at 70 F when it is zero outside.

179. Air conditioning. Among the many applications of heat and fluids that of air conditioning is attracting particular attention. A complete study of air conditioning is beyond the scope of this book, but the fundamentals are based on the principles which have been treated in this chapter and the preceding one.

Air conditioning may be defined as the simultaneous *control* of temperature, humidity, motion, and cleanliness of air. It involves: (*a*) heating or cooling (refrigeration); (*b*) humidifying or dehumidifying; (*c*) ventilation; (*d*) cleansing, washing and filtering. It is necessary in certain industries, such as printing, where the paper changes with the temperature and the moisture. It is desirable for health. Its purpose is to maintain the conditions within a *comfort zone*, that is within those limits of temperature, moisture content, and circulation in which 50 per cent or more people feel comfortable.

Moisture in the air. Air which holds all the moisture possible is saturated. Air which is saturated at one temperature can absorb more water vapor at a higher temperature, but must deposit some water vapor if cooled. The amount that it can hold at different temperatures, at saturation, is shown in tables or on a Psychrometric Chart.

Dew point is the temperature at which water vapor in the air begins to condense. Above this temperature the air is not saturated.

Relative humidity is the ratio of the amount of vapor present in air to the amount which the air could hold if saturated at that temperature. This can be found from tables or from a psychrometric chart.

A Psychrometer is a combination of dry- and wet-bulb thermometers — arranged to determine the readings in moving air. One form is called the " sling " psychrometer. Another makes use of a small fan. The *dry-bulb* thermometer gives the temperature of the air as usually noted on ordinary thermometer. The *wet-bulb* thermometer has a piece of tubular wicking immersed in water. This gives a cooler temperature, because of the evaporation from the wet-bulb, which is more rapid if the air is dry, and if the air is in motion.

Relative humidity tables give percentage of relative humidity based upon dry-bulb temperature and difference between dry- and wet-bulb temperatures. For example: If dry-bulb temperature

is 70 and wet-bulb 60, the difference is 10 and the relative humidity is 56 per cent. This " condition " is within the limits of temperature and relative humidity ordinarily considered as the comfort zone. The average of the comfort zone is about 70 deg dry-bulb and 50 per cent relative humidity. The following table gives values for a limited range. For more complete data consult a reference book, or a psychrometric chart.

RELATIVE HUMIDITY %

Difference between Dry and Wet Temperatures

Dry Bulb	5 F	6 F	7 F	8 F	9 F	10 F	11 F	12 F	13 F	14 F	15 F
62 F	74%	69%	64%	59%	54%	50%	45%	41%	37%	33%	28%
63 F	74	69	64	60	55	50	46	42	37	33	29
64 F	74	70	65	60	56	51	47	43	38	34	30
65 F	75	70	66	61	56	52	48	44	39	35	31
66 F	75	71	66	61	57	53	48	44	40	36	32
67 F	75	71	66	62	58	53	49	45	41	37	33
68 F	76	71	67	62	58	54	50	46	42	38	34
69 F	76	72	67	63	59	55	51	47	43	39	35
70 F	77	72	68	64	59	56	51	48	44	40	36
71 F	77	72	68	64	60	56	52	48	45	41	37
72 F	77	73	69	65	61	57	53	49	45	42	38
73 F	78	73	69	65	61	57	53	50	46	42	39
74 F	78	74	70	66	62	58	54	50	47	43	40
75 F	79	75	70	67	63	59	55	51	48	44	41
76 F	80	76	71	68	64	60	56	51	48	45	41
77 F	80	76	71	68	65	60	57	52	49	46	42
78 F	80	76	71	69	65	61	58	53	50	47	43
79 F	80	77	72	69	66	61	59	54	50	48	44
80 F	80	77	72	69	67	62	59	55	51	49	45

PROBLEMS

17–47. Refer to table and determine the percentage relative humidity for (*a*) dry-bulb 78 F and wet-bulb 65 F, (*b*) dry-bulb 74 F and wet-bulb 62 F, (*c*) dry-bulb 68 F and wet-bulb 57 F, (*d*) dry-bulb 62 F and wet-bulb 52 F.

17–48. What would be the wet-bulb temperature for 48 per cent relative humidity when the dry-bulb is 70 F?

SUMMARY

Heat is a form of energy due to molecular action.

Temperature is a measure of the **intensity** of heat. It is expressed in degrees on a thermometer scale. The fahrenheit scale uses 32 as the freezing point of water, and 212 as the boiling point. The centigrade uses 0 and 100 for these points.

$$F = (C \times \tfrac{9}{5}) + 32 \quad \text{and} \quad C = (F - 32) \times \tfrac{5}{9} \tag{89}$$

Absolute zero is 492 fahrenheit degrees below the freezing point of water, $= -460$ F.

$$T = t + 460 \tag{90}$$

The coefficient of expansion is the rate of expansion per unit length per degree rise.

Charles' Law is that the **volume** of a gas is directly proportional to its **absolute temperature,** provided the pressure remains constant.

$$\frac{V_1}{V_2} = \frac{T_1}{T_2} \tag{91}$$

Further, the **absolute pressure** of a gas is directly proportional to its **absolute temperature**, provided the volume remains constant.

$$\frac{p_1}{p_2} = \frac{T_1}{T_2} \tag{92}$$

The **general law of a perfect gas** is a combination of the foregoing laws with Boyle's Law, (Art. 164) and is stated as a formula in equation (93).

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2} \tag{93}$$

The British thermal unit is a measure of the **quantity** of heat. It is that quantity necessary to raise 1 lb of water 1 deg fahr.

$$H = W\,(t_2 - t_1) \tag{94}$$

Specific heat is the number of British thermal units required to raise the temperature of 1 lb of a substance 1 deg fahr. The specific heat of water is 1.

$$H = cW\,(t_2 - t_1) \tag{95}$$

The heat equation is Heat lost = heat gained.

$$c_h W_h\,(t_h - t_m) = c_c W_c\,(t_m - t_c) \tag{96}$$

The method of total heat involves an equation stating that the total heat above some base temperature is the same before and after the exchange of heat.

$$\begin{aligned} c_1 W_1\,(t_1 - t_b) + c_2 W_2\,(t_2 - t_b) + c_3 W_3\,(t_3 - t_b) = \\ c_1 W_1\,(t_m - t_b) + c_2 W_2\,(t_m - t_b) + c_3 W_3\,(t_m - t_b) \end{aligned} \tag{97}$$

Latent heat is that which is stored up in a body when there is a change of state from solid to liquid, or from liquid to gaseous. This heat is released by the reverse changes of state. The latent heat of fusion of ice is 144 Btu; of vaporization of water is 970 Btu.

Sensible heat is that which causes a change in temperature.

Thermal efficiency is the ratio of output to input in thermal units, expressed in percentage.

Heat transfer from a higher to a lower temperature level is computed by use of a coefficient called U. U is the number of British thermal units transmitted per square foot of surface per degree difference in temperature per hour. It is experimentally determined for the thickness of material as used.

$$H = AU\ (t_i - t_o) \tag{98}$$

APPENDIX A

1. Newton's three laws of motion. These fundamental laws of physics form the basis of Sir Isaac Newton's treatise on the subject, written in 1687. Because of the frequent reference to them they are re-stated here.

Law 1. *Inertia.* A body continues in a state of rest or of uniform motion in a straight line unless compelled to change that state by some external force.

Law 2. *Acceleration.* The acceleration of a body is proportional to the force which causes it.

Law 3. *Action and Reaction.* To every action there is an equal and opposite reaction.

2. Significant figures. The results of all experimental work should be so expressed as to indicate as nearly as possible the degree of precision with which the work was performed. It is evident that all numbers that are obtained as the result of measurements are limited in precision by the nature of the apparatus employed, by the care used by the observer, the size of the units, etc. In this respect, then, such quantities are quite different from the pure numbers of absolute value as employed in ordinary arithmetical operations.

The student should observe carefully the following rules in all laboratory work and in reports and problems:

I. Recording Readings

In general, scales, etc., are to be read to tenths of the smallest divisions marked on the instrument. The last figure entered in the record is thus assumed always to be an estimation and *therefore doubtful.*

Example 1.

5.57 in. means that a distance was measured by a scale subdivided to tenths of inches, and that the observer *estimated* the seven; thus the distance is known to be between 5.5 and 5.6, and *estimated* to be $\frac{7}{10}$ of the way between these two values. It is misleading, and furnishes only a clue to what is *actually known* about this distance to record it as 5.6 or 5.570 in.

Example 2.

A distance is being measured with a rule subdivided to *tenths of inches*. The observer finds the distance to be as nearly exactly 7 in. *as he can distinguish*. This should be recorded 7.00 *in.* (not 7.0 or 7 in.). Why?

Example 3.

A balance is capable of weighing an object to .01 gm and .001 gm can be estimated. Notice the correct records for the following:

Eight gm	8.000 gm
Eight and $\frac{1}{2}$ gm	8.500 gm
Eight and $\frac{7}{10}$ gm	8.070 gm
Eight and $\frac{8}{1000}$ gm	8.008 gm
Eight-tenths gm	.800 gm

In general, a series of readings made with the same instrument should all show the same number of places filled into the right of the decimal point *even if one or all these* places are *zeros*. Why?

It is often convenient to express *in decimal form* readings taken from scales divided into halves of units, quarters, eighths, etc. A table of decimal equivalents is given in Appendix B. In all such cases, retain only as many places in the decimal as correspond approximately to the same degree of precision as would be expressed by the fraction, i.e., to the nearest half unit, to the nearest quarter, etc. If the first decimal figure rejected is 5 or greater, call the preceding figure one larger than before.

A study of the following table should make this clear:

$\frac{1}{2} = .5$	$\frac{1}{8} = .1$	$\frac{3}{8} = .4$	$\frac{2}{3} = .7$
$\frac{1}{3} = .3$	$\frac{1}{16} = .06$	$\frac{5}{8} = .6$	$\frac{3}{4} = .8$
$\frac{1}{4} = .3$	$\frac{1}{32} = .03$	$\frac{7}{8} = .9$	$\frac{3}{16} = .19$
$\frac{1}{5} = .2$	$\frac{1}{64} = .02$	Etc.	Etc.
$\frac{1}{6} = .2$			

It is the practice of some engineers, when the decimal rejected is *exactly* 5, to make the next figure *even*, instead of *always larger*. This tends to compensate for differences in an extended computation. For example, 0.15 is called 0.2, 0.25 is also called 0.2, 0.35 and 0.45 are both called 0.4, etc.

II. Use of Data in Calculations

Wherever the figure following the doubtful (last retained) figure is 5 or greater than 5, increase the doubtful figure by unity.

Thus, if but three figures are to be kept, 15.75, 15.76, 15.77, 15.78, and 15.79 would all be entered 15.8.

Notice especially that the *location of the decimal point has nothing to do with significant figures.* Thus, 275, 27.5, 2.75, .275, .0275, .00275, etc., are all results expressed to the same degree of precision, and in each there are *three and only three* significant figures, the 5 being the doubtful figure in each.

Averages. In averaging a series of determinations, in general, retain in the result the same number of significant figures as in any one item.

But if a *large* number of items closely agreeing with each other are averaged, the result may contain *one more* significant figure than any item.

Multiplication. After the operation, keep in the result as many figures, *counting from the left*, as there are significant figures in the factor having the *lesser number* of significant figures.

Division. In dividing one number by another, keep in the quotient as many figures as there are significant figures in the number having the lesser number of significant figures. Continue the divisions only far enough to determine the required figures.

Note on multiplication and division. Ciphers immediately following the decimal point, *when there are no figures to the left of the point, do not count* as significant. Study the following examples:

(*a*) $15.75 \times 3.08 = 48.5$
(*b*) $.096 \times .096 = .0092$
(*c*) $.1523 \times .00113 = .000172$
(*d*) $720 \times 3.1 = 2200$
(*e*) $900 \times 800 = 720000$

In (*e*) only the *first cipher* is significant. It is necessary to add the other three to express the number properly.

(*f*) $325.6 \div 72.5 = 4.49$
(*g*) $.0007859 \div 157 = .00000506$

Use of pure numbers, constants, etc. In using pure numbers and constants such as 3.1416, .7854, etc., do not employ more figures than there are significant figures in the values depending on experiment which are used with them in the same calculation. Thus, if the diameter of a circle is measured as 4.51, the area is $4.51 \times 4.51 \times .785 = 15.9$ The use of more numbers in the constant lengthens the computation and gives no better results. Why?

3. Curves and graphs. *Coordinate axes.* The position of a point in space may be fixed by reference to two known straight lines intersecting at right angles in the same plane as the point, OX and OY of Fig. 262. Such lines are known as *coordinate axes.*

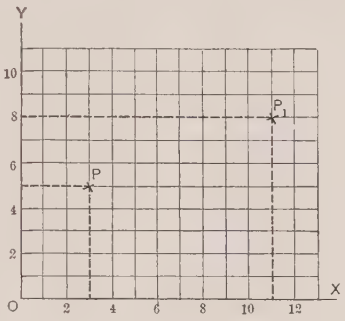


Fig. 262.

The horizontal line (OX) is known as the "*axis of abscissæ*" or "*X axis,*" the vertical line (OY) as the "*axis of ordinates*" or "*Y axis.*" The point of intersection O is called the *origin.* The *abscissa* of a point is its horizontal distance from OY; its *ordinate* is its vertical distance from OX. These given, the position of the point is determined. Thus P is that point which has an abscissa of 3, an ordinate of 5, P_1 the point which has abscissa of 11, ordinate of 8, etc.

For convenience, squared or "cross section" paper is used for work of this kind.

Curves. A succession of related points may be connected by a smooth line, thus constituting a "curve." Such curves are frequently the most convenient and the clearest way of representing a physical law, corrections for errors of apparatus, etc. Suppose, for example, that it is desired to show the relation between the stretch of a wire and the stretching loads producing it, data being as follows:

Load, Pounds	Increase in Length, Inch
5	.010
10	.019
15	.030
20	.040
25	.051

Taking the stretching loads, expressed in some convenient scale of lengths, as ordinates, and the corresponding elongations similarly expressed, as abscissæ, a series of points may be located as just explained, and through these a smooth line may be drawn. Inspection of the curve thus produced, Fig. 263, will show at a glance what could be obtained from the figures only on more extended analysis. The law, "Elongation is proportional to the load applied," is seen immediately, from the nature of the curve.

Had the curve turned continuously more and more toward either the X or the Y axis, showing in one case a progressive increase, in the other a progressive decrease in elongation with increase of load, or had, at any time, a sudden change from the conditions which had previously existed occurred, these factors would have been brought to the attention as quickly.

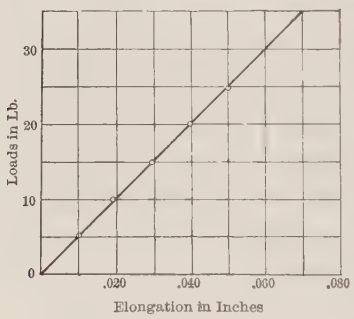


FIG. 263.

When also, as here, the great majority of points lie along a straight line (or, as in some cases, along a smooth curve), any experimental errors of measurement (as in the elongations for loads of 10 and 25 lb), will be shown at once by the fact that these points lie slightly off the line. In all such cases, the smooth curve should be drawn through as many points as possible, leaving approximately as many of the impossible points on one side as on the other.

The student must in all cases use his judgment in drawing the curve and consider the conditions of the experiment and the general physical law illustrated.

It is not necessary, and indeed often not advisable, that ordinates and abscissæ be expressed in the same scale. Of course, for the same curve all abscissæ must be in one scale, and all ordinates in one scale. In general, the scale adopted should be that most convenient for the particular values which will at the same time give a curve as large as the paper will permit.

One, two, five, or ten units to a square will be found the best. Avoid the use of three or seven units per square, or other inconvenient subdivisions.

EXERCISES

1. Locate the following points:

Point	Abscissa	Ordinate
(*a*)	5	3
(*b*)	7	10
(*c*)	5	8
(*d*)	0	12
(*e*)	5	5
(*f*)	9	0

2. Measure carefully the lengths of 9 in., 7 in., 5 in., 3 in., respectively, in metric units. Make each measurement three times, using different parts of the scale. Why? Take the average of the three readings and plot a curve, using inches as ordinates and the corresponding number of centimeters as abscissæ.

The curve will pass through the origin or zero-point of each scale. Why? From your curve find the value of 1 in. in centimeters.

What is the true value? What is your percentage of error?

3. A determination of the relation of bending of a beam to load gave following results:

Loads, Pound	Deflections, Inch
10	0.05
20	0.10
30	0.15
40	0.21
50	0.25
60	0.29
70	0.35

Plot the curve showing the relation of the deflection to load, using loads as ordinates.

4. A determination of Boyle's Law gave the following data:

Pressure	Volume of Gas
82.1	12.03
88.2	11.20
96.2	10.26
105.5	9.35
118.9	8.31
135.5	7.29
160.1	6.17

Plot the curve of the above values, using pressures as ordinates.

5. A determination showing the effect of the length of a beam upon its stiffness gave the following data:

Length, Inches	Deflection, Inch
10	.005
20	.035
30	.120
40	.285

Plot the curve showing the relation of the deflections to length, using deflections as abscissæ.

4. The equation of a straight line. It is sometimes desired to find the algebraic equation that corresponds to a given line (straight

or curved) plotted on squared paper. In this course only the *straight line* equation will be needed.

Let AB, Fig. 264, be a straight line plotted on the axes OX and OY, and meeting the Y axis at A. Let a be the vertical ordinate of A. Let x and y be the coordinates of any point P on AB. Let m be the slope of the line, = the ratio of PD to AD. Then (from analytical geometry) the equation of AB can be shown to be

$$y = a + mx$$

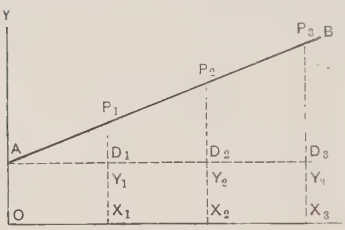


Fig. 264.

That is, the vertical ordinate of any point = the interception on the Y axis + the slope $\times$ the horizontal ordinate of the point. For example: If the line rises 4 in. for each 10 in. horizontal, the slope = 0.4 and if the point A is 2 in. above O, then *the vertical ordinate* of a point P', at a distance of 15 in. from O, $= y = 2 + 0.4 \times 15 = 8$ in.

EXERCISES

1. Plot a straight line such that it shall gain 3 units of abscissæ for every unit gained as ordinate.

2. The area of a circle varies with the square of its diameter. Plot a curve to show the relation of area to diameter in circles whose successive diameters are 1, 2, 3, 4, 5, and 6. Use diameters as ordinates.

3. What is the equation of the line which cuts the axis Y at a point 3.4 above the origin and which rises 5 units for every 8 of horizontal distance?

4. If the intercept of a line is .56 and its slope is 2.58, what is its equation?

5. A line has two points whose coordinates are (8, 5) and (3, 1). Plot the line and obtain its equation.

6. What is the slope of a line if its intercept is 4 and $x = 12$ when $y = 6.8$?

7. The slope of a certain line is .532, and it passes through the origin. What is the ordinate of a point on this line whose abscissa is 2.3?

8. Write the equation of the following lines. a = intercept and m = slope.

(*a*) $a = 5$,	$m =$	.13.
(*b*) $a = -5$,	$m =$	3.
(*c*) $a = -2.3$,	$m =$	.70.
(*d*) $a = 3$,	$m = -$	.68.
(*e*) $a = -2.5$,	$m = -$	.45.

9. Draw a sketch to show the general character of each of the lines described in Exercise 8.

5. The construction of direction and force diagrams. An angle can be determined graphically by drawing the three sides of any triangle of which it is a part. This is sometimes a more accurate method of measuring an angle on apparatus or a structure than by a protractor. In fact, seldom if ever does one observe a workman on construction using a protractor. He measures the length of three members or if that is not feasible he measures some definite distance along two of them, and the distance across space between the two points. These can be plotted to scale and the angle measured on paper with a protractor. But actually the angle need not then be measured, as the *direction* of the forces is along two sides of the triangle as plotted.

For example, in the apparatus shown in Fig. 31, it is desired to plot the directions of the three forces at O, after which to construct a force parallelogram with the forces along OA and OB as sides. It is unnecessary to determine the number of degrees in angle AOB, unless it is desired for some other reason.

The procedure is to measure the lengths of AB, AO, and BO on the apparatus. Then on paper draw a horizontal line AB to some convenient scale, large enough to have the diagram cover about half of the sheet of paper. With a compass swing arcs of the proper scaled distances from A and from B to intersect and locate the point O. From O draw a vertical line down to represent the direction of OW. Label the points, thus completing the direction diagram.

In practice the above diagram is drawn in light lines; the force diagram is then superimposed in heavy lines, to a different scale in pounds.

Example.

In the apparatus of Fig. 31, suppose $AB = 40$ in., $OA = 27$ in., and $OB = 33$ in. Also suppose the spring balance $F_1 = 30$ lb, and $F_2 = 23$ lb. Draw the direction and force diagrams, see Fig. 32.

Solution. Choose a scale of 1 in. = 8 in. for the **direction diagram.** Draw a horizontal line 5 in. long (= 40 in.) to represent AB. Draw an arc with $3\frac{3}{8}$ in. radius (= 27 in.) from A; another arc with $4\frac{1}{8}$ in. radius (= 33 in.) from B; the intersection of these arcs locates the point O. A vertical line represents the direction of OW. If the amount of the angle is desired, place a protractor on the lines as drawn, the result in this example being about 83°.

For the **force diagram,** choose a scale of 1 in. = 16 lb. Along OA

scale a distance of $1\frac{7}{8}$ in. (= 30 lb), placing an arrow head at the forward end to indicate that it is a force. Along OB scale $1\frac{7}{16}$ in. (= 23 lb), and place an arrow head at the forward end. From the forward end of each force line draw lines parallel to the opposite force to complete a parallelogram. Draw the diagonal of the parallelogram from O and scale its length in 16th inches. Its length is found to be $2\frac{1}{2} = \frac{4}{16}$ in.; therefore the resultant of the two forces = 40 lb.

The **parallelogram** is most easily constructed by the method of mechanical drawing, with T-square and triangle or with two triangles. Lay one edge of the triangle along one force line, hold the other triangle against another edge, and slide the first triangle along into position to draw a line from the forward end of the other force line. See Fig. 20. Its edge will remain parallel to its first position. Follow a similar procedure with the second line, drawing it to an intersection with the first line and thus completing the parallelogram.

6. The use of simple trigonometry. Many problems in mechanics are more readily and accurately solved mathematically than graphically, by the use of simple trigonometric functions. A brief discussion is given here, to enable the student who wishes to review, or who has not previously studied the subject, to acquire sufficient knowledge to *use* the simple formulas in solving problems in this course. For a more complete understanding of trigonometry, reference should be made to a book on the subject.

Trigonometry is based upon the right triangle: that is, a triangle in which one angle is 90°, the side opposite this 90°, or right angle being called the hypotenuse. A trigonometric function is the ratio showing the relationship of two of the sides of the right

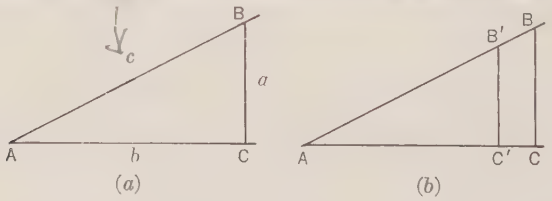


Fig. 265.

triangle, named with respect to one of the acute angles. For example, in Fig. 265(*a*) *the ratio of the length of side a to the length of side c* (= a/c) *is called the* **sine** *of angle A*. The functions most used are the sine, the cosine, and the tangent; abbreviated sin, cos, and tan, respectively.

Let A be any given angle between two lines, Fig. 265(b), and let BC be drawn from any point B on one line, perpendicular to the other line at C, to form the right triangle ABC in which one of the acute angles is the given angle A. Letter the sides a, b, c, opposite the angles with the corresponding capital letters.

Then

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c}$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c}$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b}$$

It can be shown (by geometry) that these ratios have the same value for any given angle, independent of the size of the triangle. That is, the ratio of BC to AB will have the same value as the ratio of $B'C'$ to AB', since they are similar triangles. For example, suppose that for some angle AB measures 10 units ($\frac{1}{8}$ in., or any convenient scale), and that BC measures 5.05 units. Then

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{BC}{AB} = \frac{5.05}{10} = 0.505.$$

Also suppose that AB' is $\frac{4}{5}$ as long as AB, $= 8$ units. Then, by similar triangles, $B'C'$ will be $\frac{4}{5}$ as long as BC, $= 4.04$ units, and

$$\sin A = \frac{B'C'}{AB'} = \frac{4.04}{8} = 0.505, \text{ the same as before.}$$

Note that in the same right triangle, since b is opposite $\angle B$ and adjacent to $\angle A$, $\sin B = b/c = \cos A$; $\cos B = a/c = \sin A$; $\tan B = b/a$, this function being called the cotangent (ctn) of A. Note also that, although the right triangle is the basis, the functions of any angle apply whether or not it is in a right triangle. The angle actually may be in an oblique triangle, or in no triangle, but its functions will have the same values always, being the ratios of the sides of any right triangle which can be imagined drawn to include the angle.

The values of these ratios have been computed for all angles and tabulated, as in the table on page 353. This table is carried out to three decimal places, which is precise enough for problems in this book. These functions make it possible to solve any simple right triangle (whether in units of length, velocity, or force) if two sides, or one side and an acute angle, are given.

Note that the tables give values for angles up to 90°. To obtain the values for obtuse angles, above 90°, the rules are:

The sine of an obtuse angle = sin (180° − the given angle)
The cosine of an obtuse angle = −cos (180° − the given angle)

Observe that the cosine of an obtuse angle is always negative. For example, sin 120° = sin (180° − 120°) = sin 60°; and cos 120° = −cos (180° − 120°) = −cos 60°.

Example 1.
In Fig. 265, if $\angle A = 30°$ and hypotenuse $c = 1$ in., find sides a and b.
Solution.

$$\frac{a}{c} = \sin A = \sin 30° = 0.500 \qquad \text{(see table)}$$

Therefore: $a = c \sin A = 1 \times \sin 30° = 1 \times 0.500 = 0.5$ in. *Ans.*

Also: $\frac{b}{c} = \cos A = \cos 30° = 0.866$ (see table)

Therefore: $b = c \cos A = 1 \times \cos 30° = 1 \times 0.866 = 0.866$ in. *Ans.*

Example 2.
In Fig. 265, if $\angle A = 30°$ and side $a = 1$ lb, find hypotenuse c.
Solution.

$$\frac{a}{c} = \sin 30° = 0.500$$

$$c = \frac{a}{\sin 30°} = \frac{1}{0.500} = 2 \text{ lb} \quad \textit{Ans.}$$

Example 3.
In Fig. 265, if $b = 5$ ft per sec and $\angle A = 40°$ find a.
Solution.

$$\frac{a}{b} = \tan 40° = 0.839 \qquad \text{(see table)}$$

$$a = b \tan 40° = 5 \times 0.839 = 4.195 \text{ fps} \quad \textit{Ans.}$$

Example 4.
In Fig. 265, if $\angle A = 30°$ and side $a = 1$ ft, find side b.
Solution (*a*).

$$\frac{a}{b} = \tan 30° = 0.577$$

$$b = \frac{a}{\tan 30°} = \frac{1}{0.577} = 1.73 \text{ ft} \quad \textit{Ans.}$$

Solution (*b*).

$$\frac{b}{a} = \tan 60^\circ = 1.73$$

$$b = a \tan 60^\circ = 1 \times 1.73 = 1.73 \text{ ft} \quad \textit{Ans.}$$

Example 5.

In Fig. 265, $BC = 9$ and $AC = 13$; find $\angle BAC$.

Solution.

$$\tan BAC = \frac{BC}{AC} = \frac{9}{13} = 0.692$$

From the table: Angle whose tan = 0.692 is between 34° and 35°. *Ans.*

If a more precise value is desired, it can be found by interpolation, thus:

tan 34° = 0.675, tan 35° = 0.700, which is 0.025 greater than 34°
tan BAC = 0.692, which is 0.017 greater than tan 34°

Therefore $\angle BAC$ is $\frac{17}{25}$ of a degree greater than 34°, that is,

$$\angle BAC = 34\tfrac{17}{25}^\circ = 34.68^\circ. \quad \textit{Ans.}$$

The value 34.68° can also be written 34° 40′ 48″, since 0.68° = 0.68 × 60 = 40.8 minutes, and 0.8′ = 0.8 × 60 = 48.0 seconds.

Example 6.

Referring to Fig. 25, suppose force OR to be 60 lb, and $\angle\theta$, between its line of action and the horizontal to be 36°. Find the H and V components of OR.

Solution.

Horizontal component =

$$OX = OR \cos 36^\circ = 60 \times 0.809 = 48.5 \text{ lb} \quad \textit{Ans.}$$

Vertical component =

$$OY = XR = OR \sin 36^\circ = 60 \times 0.588 = 35.3 \text{ lb} \quad \textit{Ans.}$$

Practically all problems in force analysis can be solved with no other functions than have been given. And in the majority of cases the solution can be accomplished by the application of the right triangle only. It is common practice to resolve forces acting at an angle into their components along horizontal and vertical axes, find the summation of the horizontal components and of the vertical components, and finally the resultant of these summations. This method does not involve the oblique triangle.

However, the solution of oblique triangles can be accomplished quite simply by the use of two equations, the law of sines and the

law of cosines. The development of these laws can be found in textbooks on trigonometry. As usually stated, they are:

$$\text{Law of Sines:} \qquad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\text{Law of Cosines:} \qquad c^2 = a^2 + b^2 - 2ab\cos C$$

As in the right triangle, the sides a, b, and c are opposite the angles with the corresponding capital letters.

Example 2 in Art. 28.

Two forces of 50 lb and 60 lb, respectively, act at an angle of 40°; compute their resultant.

Solution. In the *oblique* triangle, Fig. 50(*b*), $b = 60$, $a = 50$, and $\angle C = 180° - 40° = 140°$.

Then
$$c^2 = a^2 + b^2 - 2ab\cos C \qquad \text{(law of cosines)}$$
$$c^2 = 50^2 + 60^2 - 2 \times 50 \times 60 \times (-0.766) \text{ (see note)}$$
$$c^2 = 2500 + 3600 + 4596 = 10{,}696$$
$$c = \sqrt{10{,}696} = 103$$

Also
$$\frac{\sin A}{\sin C} = \frac{a}{c} \qquad \text{(law of sines)}$$
$$\sin A = \sin C \times \frac{a}{c} = 0.643 \times \frac{50}{103} = 0.312$$
$$\angle A = 18.2° \qquad \text{(see tables)}$$

Therefore the resultant is a force of 103 lb acting about 18° from the 60-lb force. *Ans.*

It may be noted that the foregoing solution for Example 2 is general for any angle, including 90° (in which case the cosine becomes zero and the last term drops out), and could be applied to the solution of Example 1 in Art. 28 as well.

In case there appears to be any confusion as to the function in the last term, when substituting in the law of cosines, above, the student should observe that $\angle C = 180° - \angle\theta$ (between the forces) $= 180° - 40° = 140°$; and, further, that cos 140° (which is greater than 90°) $= -\cos(180° - 140°) = -\cos 40° = -0.766$.

Attention is called to the fact that the Law of Cosines as treated in trigonometry differs from the formula in mechanics, in that the latter employs the $\angle\theta$ *between the forces.* $\angle\theta$ is the *external angle at C*, that is, $\angle\theta = 180° - \angle C$. Therefore $\cos\theta = -\cos C$ (since

cosines of obtuse angles are always negative). The force formula is written:

$$R^2 = a^2 + b^2 + 2\,ab\cos\theta$$

Notice the different angle and algebraic sign.

Either equation can be used, care being taken in determining the value and algebraic sign of the third term.

7. The solution of equations in algebra. A few methods of procedure are noted here for quick reference.

Simple equations can be solved by combining like terms; transposing (with sign changed) to bring the unknown quantity on one side of the equation; then dividing through by the coefficient of the unknown to determine its value. Multiplication and division must be done before addition and subtraction, unless the latter operations are grouped in parentheses, brackets, etc. Multiplying or dividing both sides of an equation by the same quantity does not destroy the equation, and furnishes a means of clearing of fractions. Likewise the same quantity can be added to, or subtracted from, both sides if desired.

Simultaneous equations with two unknowns can be solved:

(*a*) By expressing the unknown in one equation in terms of the other unknown and *substituting* in the other equation.

(*b*) By multiplying one, or both, equations by such a quantity, or quantities, that the coefficients of one of the unknowns are the same in both equations; then by the addition, or subtraction, of the two equations one unknown is *eliminated*, that is, it is canceled out, leaving a simple equation with one unknown to be determined.

Quite often simultaneous equations occur in mechanics due to moment equations in a force system. They often may be avoided by *taking the center of moments at the point of action of one, or more, of the unknown forces* and leaving one unknown in a simple equation.

Example.

Solve the simultaneous equations of Example 2, Art. 31.

Solution.

$$0.342\,C - 0.766\,T = 0 \qquad (1a)$$

$$0.940\,C - 0.643\,T - 2000 = 0 \qquad (2a)$$

Transpose (1*a*) $0.342\,C = 0.766\,T$

$$C = \frac{0.766\,T}{0.342} = 2.24\,T$$

Substitute in (2*a*) $0.940 \times 2.24\ T - 0.643\ T = 2000$

$(2.105 - 0.643)\ T = 2000$

$1.462\ T = 2000$

$T = 1370$ *Ans.*

Substituting

$C = 2.24\ T$

$C = 2.24 \times 1370$

$C = 3070$ *Ans.*

Alternate solution.

Multiply (1*a*) by 0.940, and (2*a*) by 0.342 (the coefficients of C in the opposite numbered equations).

$$0.321\ C - 0.720\ T = 0 \qquad (1b)$$

$$-0.321\ C + 0.220\ T = -684 \qquad (2b)$$

$$0.0 \quad - 0.500\ T = -684 \qquad (1b) + (2b)$$

$$T = \frac{684}{0.5} = 1370 \quad \textit{Ans.}$$

Substitute in (1*a*) $0.342\ C - 0.766 \times 1370 = 0$

$0.342\ C = 1047.9$

$C = 3070$ *Ans.*

Example 3 of Art. 31 can be solved by similar procedure. Another example of the solution of simultaneous equations is shown in the example on the shear legs in Chapter VII, Art. 46.

Quadratic equations, involving the square of an unknown quantity, are sometimes easily solved by transposing terms and *factoring.* Or such a quantity may be added to both sides of the equation as to *complete the square* in algebraic form and the equation solved by extracting the root. Or the *quadratic formula* can be used thus: If the quadratic be arranged in the form $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

APPENDIX B

SYMBOLS

A = area.

a = arm of a moment (see r); side of a triangle; linear acceleration; elementary area in a section; **a** (bold-face type) = a vector of magnitude a.

b = breadth; side of a triangle; base of a triangle; **b** (bold-face type) = a vector of magnitude b.

C = compression (stress); coefficient of expansion; degrees centigrade; circumference; a coefficient.

c = side of a triangle; hypotenuse; distance to extreme fibre; specific heat; **c** (bold-face type) = vector of magnitude c.

D = diameter.

d = depth (see h); distance in moment of inertia equation of transfer; least dimension of a column.

E = modulus of elasticity.

e = efficiency; elongation.

F = force; concentrated load; effort put into a machine; friction; degrees fahrenheit; **F** = vector of magnitude F.

f = fulcrum of a lever; coefficient of friction.

G = gravity pull (see W); gravity component, on inclined plane.

g = acceleration of gravity.

H = horizontal axis; horizontal component; total head, in fluids; total heat; **H** = horizontal vector of magnitude H.

h = height; altitude of a triangle; depth of a beam; depth of a cross-sectional area; head, in fluids.

I = moment of inertia.

J = polar moment of inertia.

K = a constant.

k = radius of gyration; coefficient of heat conductivity; a coefficient.

L = length.

M = moment of a force; bending moment.

m = mass; elementary particle of mass; slope of a line.

N = normal force or component, to a surface.

n = revolutions per unit time, as rpm.

O = point of action of concurrent forces; origin of force parallelogram; origin of coordinate axes; center or axis of moments; center of rotation.

P = power.

p = pressure; unit pressure.

Q = quantity, or rate, of discharge (fluids); volume per unit time.

q = a coefficient.

R = reaction; resultant; **R** = vector resultant.

r = radius; torque arm (see a); distance to elementary particle of mass; distance to elementary area.

s = distance; displacement; unit-stress.

T = torque; tension (stress); period of simple harmonic motion; absolute temperature.
t = thickness; time; teeth in a gear; temperature (degrees F or C).
U = coefficient of heat transfer.
V = vertical axis; vertical component; volume; shear (stress); V = vertical vector of magnitude V.
v = velocity.
W = weight, or load; force of gravity; dead load; work.
w = weight per unit length; weight per unit volume.
XYZ = coordinate axes.
x = unknown quantity; displacement in simple harmonic motion.
y = unknown quantity; deflection of a beam.
z = depth below the surface, of a liquid.
α (alpha) = angular acceleration.
δ (delta) = an angle, as of an inclined plane.
θ (theta) = angular displacement; a given angle, as between component forces.
ρ (rho) = density.
Σ (sigma) = summation of.
ϕ (phi) = a given angle; angle of friction; phase in simple harmonic motion.
ω (omega) = angular velocity.

Symbols and abbreviations in general conform to the recommendations of the American Standards Association, modified in some particular cases. The same symbol is used ordinarily for the same concept, regardless of the system of units, and regardless of the number of special values which occur, which are designated by appropriate subscripts. Letter subscripts denote values under special conditions or in special states, e.g., e = efficiency, e_m = mechanical efficiency, e_t = thermal efficiency, etc. Number subscripts denote values at designated points in an apparatus, process, or cycle, as forces at points 1, 2, and 3 are designated by F_1, F_2, and F_3. Vectors are represented by printing in bold-face type, thus vectors of magnitude F_1, F_2, and F_3, are represented $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$. If the vector is named by the letters at each end, they are printed in italic type. If only the magnitude, or length, of a vector is indicated, as in a formula in which the direction is given by an angle, the letter is printed in italic type.

USEFUL NUMBERS

$\pi = 3.1416 \qquad \frac{\pi}{4} = 0.7854 \qquad \pi^2 = 9.8696$

Circumference of circle $= 2\pi r = \pi D = 3.1416\, D$

Area of circle $= \pi r^2 = \frac{\pi D^2}{4} = 0.7854\, D^2$

Area of triangle $= \frac{1}{2}$ base $\times$ altitude $= \frac{bh}{2}$

Surface of cylinder $= 2\pi rL + 2\pi r^2$
Volume of cylinder $= \pi r^2 L$
Surface of sphere $= 4\pi r^2 = \pi D^2$

Volume of sphere $= \frac{4\pi r^3}{3} = \frac{\pi D^3}{6} = 0.5236\ D^3$

1 cu ft = 1728 cu in. = 7.48 gal
1 gal = 231 cu in.
1 lb = 7000 grains
1 mile = 5280 ft
1 mph = 1.467 fps
1 fps = 0.682 mph
1 hp = 33,000 ft-lb per min = 550 ft-lb per sec
1 hp = 746 watts
1 kw = 1000 watts = 1.34 hp
1 watthour = 3.4 Btu
1 Btu = 1055 watt-second = 778 ft-lb
Latent heat of fusion of ice = 144 Btu
Latent heat of vaporization of water = 970 Btu
1 gal of water weighs about 8.33 lb
1 cu ft of water weighs about 62.5 lb (1000 oz)
1 cu in. of water weighs about 0.0361 lb
1 cu ft of air (standard) weighs about 0.0807 lb
1 cu ft of air at 70 F weighs about 0.0749 lb

TABLE OF MATERIALS

Material	Density Per Cu Ft	Density Per Cu In.	Specific Gravity	Specific Heat	Coefficient Linear Expansion
Aluminum	166	.096	2.67	.219	.0000123
Asbestos	153	.088	2.45	.195	.....
Brass	532	.308	8.55	.094	.0000105
Brick	125	.072	2.00	.220	.0000030
Cement	90	.052	1.45	.270	.0000060
Concrete	145	.084	2.43	.270	.0000079
Copper	556	.322	8.93	.095	.0000088
Cork	15	.008	0.24	.485	.....
Glass	164	.094	2.60	.198	.0000439
Iron (cast)	450	.260	7.21	.130	.0000055
Iron (wrought)	485	.287	7.78	.114	.0000067
Lead	710	.410	11.35	.031	.0000157
Marble	170	.098	2.73	.210	.0000065
Mercury	848	.491	13.60	.033	.....
Steel	488	.283	7.83	.117	.0000065
Stone (granite)	160	....	2.78	.195	.0000048
Stone (sandstone)	144	....	....	....	.....
Stone (slate	175	....	....	....	.....
Timber (oak)	45	.026	0.72	.570	.0000028
Timber (soft pine)	25	.014	0.40	.670	.0000027
Wood (lignum vitæ)	...	.048	1.33	....	.....

Fraction-Decimal Equivalents

$\frac{1}{32}$	.031	$\frac{17}{32}$	.531
$\frac{1}{16}$	.062	$\frac{9}{16}$	.562
$\frac{3}{32}$	.094	$\frac{19}{32}$	.594
$\frac{1}{8}$	.125	$\frac{5}{8}$	.625
$\frac{5}{32}$	.156	$\frac{21}{32}$	.656
$\frac{3}{16}$	.188	$\frac{11}{16}$	.688
$\frac{7}{32}$	.219	$\frac{23}{32}$	.719
$\frac{1}{4}$	.250	$\frac{3}{4}$	.750
$\frac{9}{32}$	.281	$\frac{25}{32}$	.781
$\frac{5}{16}$	.313	$\frac{13}{16}$	.812
$\frac{11}{32}$	.344	$\frac{27}{32}$	.844
$\frac{3}{8}$	.375	$\frac{7}{8}$	.875
$\frac{13}{32}$	.406	$\frac{29}{32}$	.906
$\frac{7}{16}$	.438	$\frac{15}{16}$	.938
$\frac{15}{32}$	.469	$\frac{31}{32}$	.969
$\frac{1}{2}$	.500		

Metric Units and Equivalents

1 cm	= .39 in.	1 in.	= 2.54 cm
1 m	= 39.37 in.	1 ft	= 30.48 cm
1 m	= 3.23 ft	1 ft	= .305 m
1 km	= .6 mile	1 mile	= 1.60 km
1 gm	= .035 oz (avoir.)	1 oz	= 28.35 gm
1 kgm	= 2.204 lb (avoir.)	1 lb	= 453.6 gm
1 sq cm	= .154 sq in.	1 sq in.	= 6.45 sq cm
1 cu cm	= .061 cu in.	1 cu in.	= 16.39 cu cm

1 liter = 1000 CC = 1.0567 qt (liquid)

1 cu cm of water weighs ..... 1.0 gm

1 cu cm of air ("Standard") weighs ..... .00129 gm

1 dyne = .00102 gm

1 poundal = 1.3825 dynes

1 ft-lb = 1.356×10^7 ergs

1 joule = 10^7 ergs

1 watt = 10^7 ergs per sec = 1 joule per sec

1 horsepower = 7.46×10^9 ergs per sec

1 Btu = 252 calories

1 gm water heated 1 C = 4.2×10^7 ergs

TRIGONOMETRIC FUNCTIONS

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
0	.000	1.000	.000	....	....	....	....
1	.017	.999	.017	46	.719	.695	1.04
2	.035	.999	.035	47	.731	.682	1.07
3	.052	.999	.052	48	.743	.669	1.11
4	.070	.998	.070	49	.755	.656	1.15
5	.087	.996	.087	50	.766	.643	1.19
6	.105	.995	.105	51	.777	.629	1.23
7	.122	.993	.123	52	.788	.616	1.28
8	.139	.990	.141	53	.799	.602	1.33
9	.156	.988	.158	54	.809	.588	1.38
10	.174	.985	.176	55	.819	.574	1.43
11	.191	.982	.194	56	.829	.559	1.48
12	.208	.978	.213	57	.839	.545	1.54
13	.225	.974	.231	58	.848	.530	1.60
14	.242	.970	.249	59	.857	.515	1.66
15	.259	.966	.268	60	.866	.500	1.73
16	.276	.961	.287	61	.875	.485	1.80
17	.292	.956	.306	62	.883	.469	1.88
18	.309	.951	.325	63	.891	.454	1.96
19	.326	.946	.344	64	.898	.438	2.05
20	.342	.940	.364	65	.906	.423	2.14
21	.358	.934	.384	66	.914	.407	2.25
22	.375	.927	.404	67	.921	.391	2.36
23	.391	.921	.424	68	.927	.375	2.48
24	.407	.914	.445	69	.934	.358	2.61
25	.423	.906	.466	70	.940	.342	2.75
26	.438	.898	.488	71	.946	.326	2.90
27	.454	.891	.510	72	.951	.309	3.08
28	.469	.883	.532	73	.956	.292	3.27
29	.485	.875	.554	74	.961	.276	3.49
30	.500	.866	.577	75	.966	.259	3.73
31	.515	.857	.601	76	.970	.242	4.01
32	.530	.848	.625	77	.974	.225	4.33
33	.545	.839	.649	78	.978	.208	4.70
34	.559	.829	.675	79	.982	.191	5.14
35	.574	.819	.700	80	.985	.174	5.67
36	.588	.809	.727	81	.988	.156	6.31
37	.602	.799	.754	82	.990	.139	7.12
38	.616	.788	.781	83	.993	.122	8.14
39	.629	.777	810	84	.995	.105	9.51
40	.643	.766	.839	85	.996	.087	11.43
41	.656	.755	.869	86	.998	.070	14.30
42	.669	.743	.900	87	.999	.052	19.08
43	.682	.731	.933	88	.999	.035	28.64
44	.695	.719	.966	89	.999	.017	57.28
45	.707	.707	1.000	90	1.000	.000	Infinity

$$\sin \angle = \frac{\text{opposite side}}{\text{hypotenuse}} \quad \cos \angle = \frac{\text{adjacent side}}{\text{hypotenuse}} \quad \tan \angle = \frac{\text{opposite side}}{\text{adjacent side}}$$

For angle greater than 90°, $\sin \angle = \sin (180° - \angle)$, $\cos \angle = -\cos (180 - \angle)$, $\tan \angle = -\tan (180 - \angle)$.

FUNCTIONS OF NUMBERS. 1–50

Number or Diameter	Square	Square Root	Circular Area	Circumference
$\frac{1}{8}$	.0156	.3535	.0123	.3927
$\frac{1}{4}$	.0625	.5000	.0491	.7854
$\frac{1}{2}$	.2500	.7071	.1963	1.5708
$\frac{3}{4}$	.5625	.8660	.4417	2.3562
1	1	1.000	.7854	3.1416
2	4	1.414	3.1416	6.2832
3	9	1.732	7.0686	9.4248
4	16	2.000	12.566	12.566
5	25	2.236	19.635	15.708
6	36	2.449	28.274	18.849
7	49	2.645	38.484	21.991
8	64	2.828	50.265	25.132
9	81	3.000	63.617	28.274
10	100	3.162	78.540	31.416
11	121	3.316	95.033	34.558
12	144	3.464	113.097	37.699
13	169	3.605	132.732	40.841
14	196	3.741	153.938	43.982
15	225	3.873	176.715	47.124
16	256	4.000	201.062	50.265
17	289	4.123	226.980	53.407
18	324	4.242	254.469	56.549
19	361	4.358	283.529	59.690
20	400	4.472	314.160	62.832
21	441	4.582	346.361	65.973
22	484	4.690	380.133	69.115
23	529	4.795	415.476	72.257
24	576	4.899	452.390	75.398
25	625	5.000	490.875	78.540
26	676	5.099	530.930	81.681
27	729	5.196	572.555	84.823
28	784	5.291	615.752	87.965
29	841	5.385	660.520	91.106
30	900	5.477	706.858	94.248
31	961	5.567	754.768	97.389
32	1024	5.656	804.248	100.531
33	1089	5.744	855.300	103.673
34	1156	5.831	907.920	106.814
35	1225	5.916	962.113	109.956
36	1296	6.000	1017.876	113.097
37	1369	6.082	1075.210	116.239
38	1444	6.164	1134.115	119.381
39	1521	6.245	1194.591	122.522
40	1600	6.324	1256.637	125.664
41	1681	6.403	1320.254	128.805
42	1764	6.480	1385.442	131.947
43	1849	6.557	1452.201	135.088
44	1936	6.633	1520.531	138.230
45	2025	6.708	1590.431	141.372
46	2116	6.782	1661.902	144.513
47	2209	6.855	1734.944	147.655
48	2304	6.928	1809.557	150.796
49	2401	7.000	1885.741	153.938
50	2500	7.071	1963.500	157.080

FUNCTIONS OF NUMBERS. 51–100

Number or Diameter	Square	Square Root	Circular Area	Circumference
51	2601	7.141	2042.8	160.221
52	2704	7.211	2123.7	163.363
53	2809	7.280	2206.2	166.504
54	2916	7.348	2290.2	169.646
55	3025	7.416	2375.8	172.788
56	3136	7.483	2463.0	175.929
57	3249	7.550	2551.8	179.071
58	3364	7.616	2642.1	182.212
59	3481	7.681	2734.0	185.354
60	3600	7.746	2827.4	188.496
61	3721	7.810	2922.5	191.637
62	3844	7.874	3019.1	194.779
63	3969	7.937	3117.2	197.920
64	4096	8.000	3217.0	201.062
65	4225	8.062	3318.3	204.204
66	4356	8.124	3421.2	207.345
67	4489	8.185	3525.7	210.487
68	4624	8.246	3631.7	213.628
69	4761	8.307	3739.3	216.770
70	4900	8.367	3848.5	219.911
71	5041	8.426	3959.2	223.053
72	5184	8.485	4071.5	226.195
73	5329	8.544	4185.4	229.336
74	5476	8.602	4300.8	232.478
75	5625	8.660	4417.9	235.619
76	5776	8.718	4536.5	238.761
77	5929	8.775	4656.6	241.903
78	6084	8.832	4778.4	245.044
79	6241	8.888	4901.7	248.186
80	6400	8.944	5026.5	251.327
81	6561	9.000	5153.0	254.469
82	6724	9.055	5281.0	257.611
83	6889	9.110	5410.6	260.752
84	7056	9.165	5541.8	263.894
85	7225	9.220	5674.5	267.035
86	7396	9.274	5808.8	270.177
87	7569	9.328	5944.7	273.319
88	7744	9.381	6082.1	276.460
89	7921	9.434	6221.1	279.602
90	8100	9.487	6361.7	282.743
91	8281	9.539	6503.9	285.885
92	8464	9.592	6647.6	289.027
93	8649	9.644	6792.9	292.168
94	8836	9.695	6939.8	295.310
95	9025	9.747	7088.2	298.451
96	9216	9.798	7238.2	301.593
97	9409	9.849	7389.8	304.734
98	9604	9.899	7543.0	307.876
99	9801	9.950	7697.7	311.018
100	10000	10.000	7854.0	314.160

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